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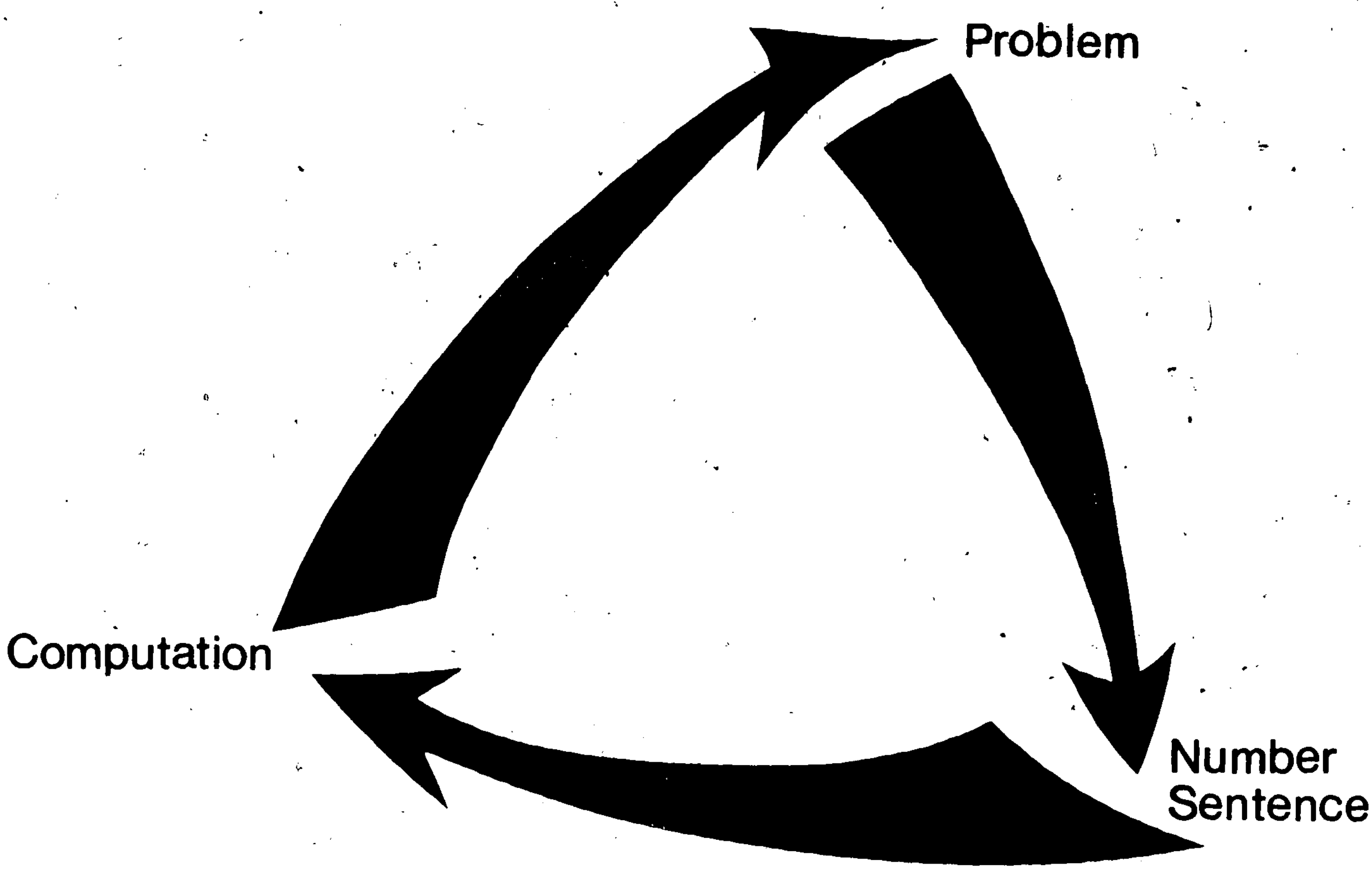
ABSTRACT

This guide is organized around six concepts: sets, numbers and numeration; operations, their properties and number theory; relations and functions; geometry; measurement; and probability and statistics. Objectives and sample activities are presented for each concept. Separate sections deal with the processes of problem solving and computation. A section on updating curriculum includes discussion of continuing program improvement, evaluation of pupil progress, and utilization of media. (MP)

mathematics for georgia schools

VOLUME II UPPER ELEMEN[illegible]Y GRADES



Charles McDaniel • State Superintendent of Schools • Georgia Department of Education • Reprint 1978

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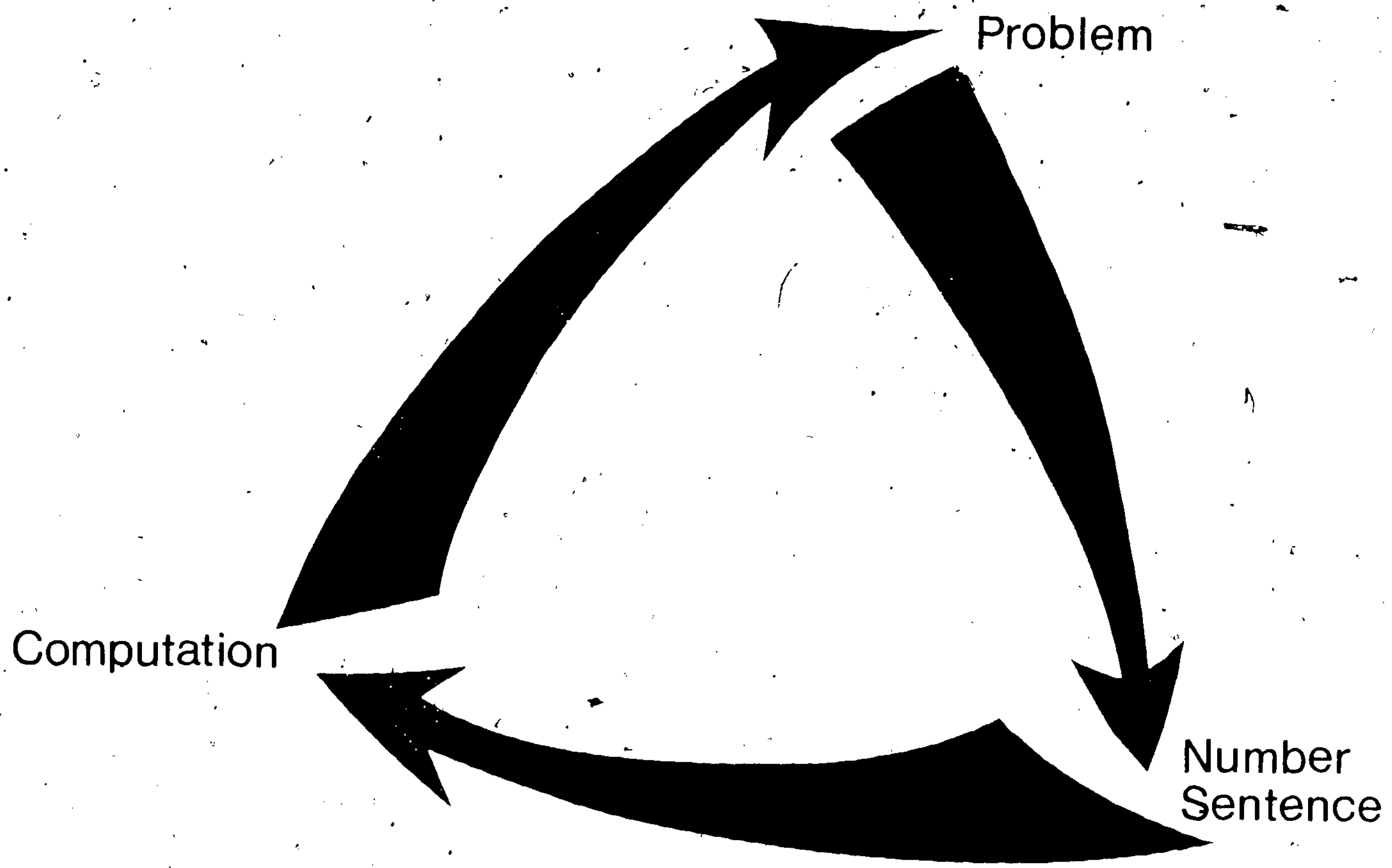
VOLUME II UPPER ELEMENTARY GRADES



Division of Curriculum Development and
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FOREWORD

The Georgia Department of Education is constantly alert to the curricular changes which seem desirable as a result of studies and experiments in various fields. A committee was appointed in 1969 to rewrite the mathematics curriculum guides for the elementary schools incorporating findings from current curriculum studies in mathematics.

This committee was composed of rural and city public school teachers and supervisors, college teachers and one out-of-state consultant who is nationally known in mathematics education. They looked at the nation's best programs in mathematics education. They considered creative ideas of teaching which fit the age of space but which are as fundamental as adding two and two. They recognized that mathematics is an essential part of life itself and is a daily necessity for all people. The committee, taking the position that mathematics instruction is a process of initiating and nurturing understanding, felt that it would be necessary to discover techniques for accommodating the differing rates at which children develop.

LETTER TO UPPER ELEMENTARY GRADE TEACHERS

This guide has been written to assist you in improving the teaching of mathematics in the upper elementary grades. The committee and I have worked diligently for several years preparing this material and trust that the format is arranged so that it will be useful to you.

The guide has been organized around six central concepts called strands. They are entitled (1) Sets, Numbers and Numeration, (2) Operations, Their Properties and Number Theory, (3) Relations and Functions, (4) Geometry, (5) Measurement, (6) Probability and Statistics. These strands include the major mathematical concepts which undergird an updated mathematics program for children. The concepts are threads running through the curriculum and are expanded and enlarged in a spiral approach.

Each strand is introduced in terms of broad performance objectives which the teacher can make more specific by adapting them to the needs of particular children. There are one or more activities keyed to each objective. The list of objectives for each strand is placed at the end of the strand on a fold-out sheet. This allows the teacher to view the objectives as he selects activities to implement specific objectives. These activities are not sufficient to achieve the objectives. They are suggestions of kinds of experiences which will help reach the objectives.

The strands on Probability and Statistics and on Relations and Functions are included particularly because of new ideas in elementary school mathematics. It is hoped that teachers will accept the challenge of new topics, different approaches and experimental activities as a means of extending the spiral learning of mathematics for all pupils according to their potential.

There are separate sections in the guide which deal specifically with processes. Problem solving is considered a part of all mathematics and therefore is emphasized in a cross-strand approach. Computation, also, is viewed by the committee as permeating all strands, and the related section is intended to give detailed development for especially difficult procedures. Problem solving is *thinking through*, and computation is the manipulation of various symbols and terms used to express these thoughts.

Other sections are included to facilitate use of the guide by the teacher. While not prescriptive, the content and methods identified throughout the guide are of increasing importance in a contemporary mathematics program. The section on media lists instructional aids, and the use of aids is suggested in the activities of each strand. The correct use of the materials will help in the achievement of the objectives. Teachers should realize the importance of teaching children correct vocabulary and correct use of symbols. A glossary for the teacher is included to provide definitions which can be simplified into children's language. Words often used in daily communication, particularly some geometric terms, have a different meaning when considered mathematically. Symbols are to be understood as a means of stating problems and recording results after meaningful experiences with physical models of mathematical principles.

The teacher, guided by the objectives in each strand, should endeavor to determine those topics and activities most appropriate for realizing the objectives for the particular children being taught and should correlate these ideas with those in texts and other available materials. After a strand has been presented, the teacher should evaluate in terms of the objectives using instruments constructed for this purpose. Sample instruments are included in the Evaluating Pupil Progress section. In the bibliography are suggested materials designed to help implement achievement of the objectives. Early selection and purchase of materials for the library, a grade level or an individual class will insure access to books when needed.

Inservice programs for those who need help with the new ideas will result in a more competent faculty as well as increased knowledge on the part of the children. Assistance in improving local programs may be found in the section Continuing Program Improvement.

Gladys M. Thomason

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POINT OF VIEW

Curriculum planning is a continuous process of updating content, improving methods, analyzing objectives, measuring learning and appraising attitudes. The guide, *Mathematics for Georgia Schools,* is to help local curriculum committees and teachers of mathematics to identify the content, procedures and materials which will strengthen and enrich the mathematics educational program for the elementary school children of Georgia, and to measure the effectiveness of the program.

In the guide objectives are stated in behavioral terms. Local curriculum committees may find it helpful to state more specific objectives. The activities support the theory that learning is experiencing. The objectives and activities are organized into six strands written for primary grades and upper grades.

The ordering of the strands in the guide does not imply the ordering of presentation of subject matter; that is, one strand need not be completed (or even begun) before proceeding to another. The volume of material on different topics does not imply that one is more important than the other. Topics especially difficult to present and those not generally covered in currently available textbooks are developed in more detail. Individual teachers will need to make appropriate choices according to the needs of their pupils.

One strand which has emphasis is Relations and Functions, since most of mathematics involves relations between numbers and/or geometric figures. Since relations and functions are unifying concepts in mathematics, children should be encouraged to think in terms of them.

The guide does not restrict geometry to naming shapes and measuring them. Emphasis in geometry is placed on the relations between point sets such as *has the same shape, parallel to* and *congruent to.* The activities enable children to work with materials in order to learn these relations for themselves.

The emphasis on sets in this guide endorses the concept that the language of sets is a powerful tool in communicating mathematical ideas and can be used both to organize and describe.

Evaluation is a continuous and integral part of the successful elementary school program. The techniques of evaluation must include procedures for appraising interests and attitudes as well as skills and understanding.

Perhaps the one factor most essential to the success of the mathematics curriculum is understanding. To promote understanding a distinction is made between operations and computations. An operation is an assignment of a single number to an ordered pair of numbers. Computation is a process of manipulation of numerals by which one determines a name of the single number assigned to the ordered pair of numbers. The need to find a more efficient and enlightening method of instruction has led to the conclusion that clear understandi of essential mathematics concepts must precede, but certainly not supplant, the traditional point of emphasis, computation.

If mathematics instruction is viewed as a process of initiating and nurturing understanding, it will be necessary to discover techniques for accommodating the different development rates of children. Children develop concepts of mathematics from their experiences with physical objects. This guide is designed to help the teacher exercise professional judgment in adopting a mathematical program compatible with each child's ability.

CONCEPTS ACCORDING TO STRANDS

Sets, Numbers and Numeration

Operations, Their Properties and Number Theory

Relations and Functions

Geometry

Measurement

Probability and Statistics

SETS, NUMBERS AND NUMERATION

INTRODUCTION

The concept of a set is a useful tool in the study of mathematics, and the language of sets enables one to communicate mathematical ideas with clarity and precision. In the upper elementary grades many different kinds of sets are studied, for instance, sets of points in geometry, sets of equivalent fractions in the development of the rational number concept and sets of factors in number theory. Activities should be selected which will help pupils develop understandings and skills necessary to identify, describe and classify sets.

Whole numbers may be defined as properties of finite sets; more precisely, a whole number is an abstract concept associated with a class of equivalent (finite) sets. For instance, the number 5 is the common property of all sets which can be put into one-to-one correspondence with the set of fingers on one hand. Counting is the process of assigning a whole number to a finite set. Activities such as those suggested in the related strand for the primary grades lay the foundations for understanding number and for learning the system of symbols for denoting numbers. In the upper grades, the concepts of whole number and numeration are extended, with emphasis on the place value principle used in writing numerals. Work with numeration systems other than the decimal system is included in order to reinforce understanding of place value – an understanding essential for developing computational skills.

Certain applications of whole numbers lead to the important concept of an *ordered pair of numbers*. For instance, the whole numbers 2 and 5 are components of the ordered pairs symbolized in the following examples.

(a) $\frac{2}{5}$ (read "2 for 5"), where the ordered pair expresses the rate of 2 balloons for 5 cents.

(b) 2/5 (read "2, 5"), where the ordered pair expresses the date, February 5th.

(c) (2, 5) (read "2, 5"), where the ordered pair is a member of the solution set for the open sentence $\square + 3 = \triangle$.

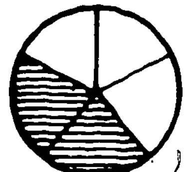
(d) (2, 5) (read "2, 5"), where the ordered pair is associated with a point in the coordinate plane.

(e) $\frac{2}{5}$ (read "2 to 5"), where the ordered pair expresses the ratio of the number of holidays to the number of school days ub a particular week.

All of the above situations involving ordered pairs of whole numbers should be dealt with in the elementary school. (See the strand on Relations and Functions for some suggested activities.) It should be noted that the symbol for an ordered pair has meaning only in terms of the context in which the ordered pair is used.

The ordered pair (2,5) is studied in still another context in elementary school mathematics, the *fraction context*. In that case, 5 is the count of parts into which a unit or a set of some kind has been partitioned, and 2 is the count of those parts which have been singled out or marked for attention, as in the following illustrations.

(2 parts out of 5 parts in a unit strip)



(2 parts out of 5 parts in a unit disc)

(2 balls out of 5 balls in a unit set)

The symbol for the number pair used in the fraction context is generally written as $\frac{2}{5}$ and read "2 over 5" or "2 fifths."

Use of the term *rational number* for the number pair in this case should be postponed until pupils reach a level of mathematical maturity sufficient to understand that a rational number is an equivalence class. For instance, the infinite set of ordered pairs,

$$\left\{ \frac{2}{5}, \frac{4}{10}, \frac{6}{15}, \frac{8}{20}, \frac{10}{25}, \frac{12}{30}, \ldots \right\}$$

represents one (exactly one) rational number.

There are several numeral forms which the upper elementary school pupil learns to use in expressing fractions (or rational numbers).

(1) The fraction (or ratio-like) form, e.g., $\frac{13}{5}$ (read "13 over 5" or "13 fifths").

(2) The decimal numeral form, e.g., 2.6 (read "2 and 6 tenths" or "26 tenths").

(3) The mixed numeral form, e.g., $2\frac{3}{5}$ (read "2 and 3 fifths").

It should be noted that what one has often called *decimal numbers* or *mixed numbers* are, in fact, fractions (or rational numbers) expressed in decimal numerals or in mixed numerals. It is important for pupils to know that the symbols $\frac{13}{5}$, 2.6, $2\frac{3}{5}$ all represent exactly the same number and that a preference for one of the numerals depends, generally, on the application or use one makes of the number.

In addition to the study of the whole numbers and rational numbers, there is another kind of number which is an important part of the developing concept of number, that is, the integers. As with many of the topics in the guide, there is no particular time or place in the curriculum when the study of integers should be initiated or completed. Certain informal uses of negative numbers or readiness experiences with signed numbers can occur as early as the primary grades. Many pupils have experiences with temperatures above and below zero and with gains and losses in football yardage statistics or other game-related scores. In working with a numberline some pupils may wonder about the numbers on the other side of zero. Such experiences or ideas should be built on and expanded throughout the grades. The suggested activities in this strand are restricted to the development of the concept of integers as *numbers*. The concept of integers as elements of a number system is in the strand on Operations, Their Properties and Number Theory.

SETS, NUMBERS AND NUMERATION

Objectives Keyed to Activities | **ACTIVITIES**

Sets

obj. 1 — 1. Have the pupils name some sets of objects in the room such as the set of desks, set of pupils, etc.

obj. 2, 3, 4 — 2. Have the pupils name sets of groups of which they are a part.

Examples

Set of pupils who attend ______________________ school.

Set of pupils who ride the bus.

Set of pupils who bring their lunch.

Let each pupil read aloud one of his categories and have those who are members stand. Also let the ones who are not members of each set stand to represent the complement of the given set.

If a pupil does not name a set that is not well defined, the teacher may describe a set such as "the set of pupils who are wearing new dresses" or "the set of pupils who talk too loudly in the lunchroom." The pupils will easily see that these sets are not well defined because "new" and "loudly" are not specific.

obj. 2,3,4 — 3. Ask those who are members of the set of pupils over 25 years of age to stand. Name and discuss other examples of the empty set.

obj. 1,2, 3,4 — 4. Use examples from English lessons to reinforce the concept of sets.

Examples

(a) Let A = {there, thought, which, thrown, thinned, talked, whose, torn, wilted, thorn}, or some other set of spelling words. In assigning this particular spelling list, ask the pupils to find the subset of words which begin with *th* and the subset of words which ends in *ed*. They may write B = {there, thrown, thought, thinned, thorn} and C = {talked, thinned, wilted}.

Then the pupils might be directed to find $B \cap C$, the set of words which begin with *th* and end in *ed*. They should find that $B \cap C = \{\text{thinned}\}$. Other subsets which they could be asked to tabulate might include the subset of words beginning with *wh*; the subset of words which can be used as a noun or the subset of words which represent the past tense of a verb.

(b) Another use of set language in studying the English lesson might be as follows.

Given a set of sentences from which one is to pick out all modifiers, have pupils tabulate the following.

A = the set of all modifiers

B = the set of all clause modifiers

C = the set of all phrase modifiers

D = the set of all word modifiers

E = the set of all objectival modifiers

F = the set of all adverbial modifiers

After the pupils have tabulated these, they may be asked to tabulate $B \cap E$, $B \cap F$, $C \cap D$ and the like.

obj. 1,3

5. To teach the descriptive method of designating a set, the pupils could be asked to describe sets such as the following.

$$A = \{10, 100, 1000, \ldots\}$$
$$B = \{10, 20, 30, \ldots\}$$
$$C = \{a, e, i, o, u\}$$

When they describe the sets as

A is the set of powers of 10,

B is the set of multiples of 10 and

C is the set of vowels,

they will not only have had practice in using the descriptive method but also in observing the common property of given sets.

obj. 2,4

6. Have the pupils consider two different sets of numbers.

Let A = set of whole numbers
B = set of integers

Have the pupils name many subsets of the two sets such as $\{x \mid x \text{ is a number} < 3\}$ observing that if this set is a subset of A it is a different set from the set that is a subset of B. Also have the pupil examine subsets of the two given sets that would be the same, such as $\{x \mid x \text{ is a number} > 3\}$. Activities such as these give the students some experience with finite and infinite sets.

obj. 2,4

7. Give the pupil opportunities to find subsets of an infinite set of numbers. For example, let A be the set of whole numbers.
Have the pupils list certain subsets such as the following.

B = the set of even numbers

C = the set of multiples of 5

D = the set of factors of 12

E= the set of numbers < 3

F = the set of numbers > 6

G = the set of numbers > 3 and ≤ 6

H = the set of numbers < 3 and ≥ 6

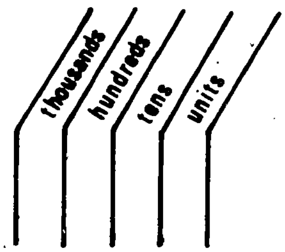
I = the set of numbers such that $2N = 6$

J = the set of numbers such that $N + 6 < 10$

Whole Numbers

obj. 5,6

8. Review the place value code for writing numerals and extend the idea that place value names are a convenient way of identifying the number of subsets that can be made by grouping objects into subsets of ten. Stress that the convention for writing numerals requires that the count of tens be recorded to the left of the count of the units. The relationship between places is ten times the one to the right. The place value chart is a good device to illustrate this convention and can serve as readiness for developing exponential notation.



The teacher will need to provide experiences with physical objects to build concepts of place value for those pupils who have not acquired them by the end of the primary grades. (See the activities in guide for primary grades.)

obj. 6 9. Activities should be used to develop skills in writing numerals in expanded notation and exponential notation as exemplified here.

$$1235 = 1(1000) + 2(100) + 3(10) + 5(1)$$
$$1235 = 1(10^3) + 2(10^2) + 3(10^1) + 5(1)$$

obj. 7 10. Number bases other than base ten may be presented to strengthen the pupil's understanding of the place value scheme of base ten. In presenting a *new* base, the teacher needs to follow the same basic steps as are followed in presenting base ten. Physical objects should be used first and then pictures, drawings and color representations before moving completely to abstractions. Most students need to be guided carefully through these steps. The examples which follow represent these steps.

Use artificial fruit, 1-inch squares, plastic tops or toothpicks, and let each single object represent *one*. In presenting base five, place five of the objects within a boundary and let this set represent *five*. Each of the sets representing *five* may be made by placing five objects in a plastic bag or placing a rubber band around five toothpicks. To represent *twenty-five* place five groups of *five* in another plastic bag. In this organization one, five, and twenty-five are represented as 1_{five}, 10_{five}, and 100_{five}. Counting and representing the numbers with objects which have been *bundled* in this way helps the pupil to understand the place value scheme. As the pupil counts by ones he must bundle or group the counters when he reaches the numbers five, twenty-five, etc. He can place the representations and record them on a large sheet of paper or board which has been blocked or marked in place value columns.

From representations using bundling the pupil would progress to those of a more abstract nature such as color representations. For example, yellow strips might represent ones, red strips fives, and green strips five fives. Such a scheme is helpful to the pupil as he begins computation. For example, to add 22_{five} and 14_{five} the pupil would represent 22_{five} with two red strips and two yellow strips and would represent 14_{five} with one red strip and four yellow strips. Combining the two groups he would get three red strips and six yellow strips; however, working in base five he must exchange five of the yellow strips for one red strip so that the result would be four red strips and one yellow strip which would be recorded as follows.

22_{five}	2 reds	2 yellows
14_{five}	1 red	4 yellows
41_{five}	~~3 reds~~	~~6 yellows~~
	4 reds	1 yellow

For further explanation and similar activities, see the strand Operations, Their Properties and Number Theory and the section Difficulties in Computation.

Before such activities are presented the pupil should practice using the colored strips to represent numbers such as 123_{five} and he should practice recording the numerals in base five to represent a set of strips such as three greens, four reds and two yellows.

After the pupil is able to represent or record numbers in different bases he should then learn to convert numerals recorded in one base to numerals in another base by using expanded notation.

Example

Let $234_{five} = x_{ten}$

$234_{five} = 2$ (five fives) + 3 (fives) + 4 (ones) or

$234_{five} = 2(100_{five}) + 3(10_{five}) + 4(1_{five})$

$x_{ten} = 2(25_{ten}) + 3(5_{ten}) + 4(1_{ten})$ Since $100_{five} = 25_{ten}$ and $10_{five} = 5_{ten}$.

$x_{ten} = 50_{ten} + 15_{ten} + 4_{ten}$

$x_{ten} = 69_{ten}$

Then $234_{five} = 69_{ten}$

Some valuable situations can be derived by having the pupil make his own symbols and names for numbers. In the construction of the system he will see that a finite number of symbols will be needed according to the number base he uses.

obj. 6,7

11. Convert numerals written in another base into base ten numerals by using expanded notation.

Example

$113_{five} = x_{ten}$

$112_{five} = 1(\text{five} \times \text{five}) + 1(\text{five}) + 2(\text{one})$

$112_{five} = 1(10^2) + 1(10^1) + 2(5)^0$

$x_{ten} = 25 + 5 + 2 = 32$

obj. 7

12. As an outgrowth of social studies, pupils can see a numeration system as another way man has of communicating his ideas (number). By looking at other systems of numeration (i.e. the Babylonian or Egyptian system) they can compare them to their own system and gain a better appreciation of the Hindu-Arabic system.

obj. 8a

13. Choose some physical model of a unit, for instance a circular cardboard disc or a rectangular sheet of paper, and say to the class

"I am going to cut across this disc like this and then this," or

"I am going to fold this paper this way and then that way."

After partitioning the unit model by cutting or folding so that there are four pieces of the same size, ask, "How many pieces do you see now?"
When pupils have agreed on the number of pieces or subregions, record *4* on the board. Then have a pupil shade three pieces of the disc or have a pupil color three of the paper subregions determined by fold lines and ask, "How many pieces of the disc did John shade?" or "How many of the paper subregions did Susan color?"
In recording their response, position the numeral *3* so that the two numerals look like this.

3, 4 (read "three comma four") or $\frac{3}{4}$ (read "three over four").

The teacher should point out, "This pair of whole numbers describes the experiment we have just done." "The number 4 is a count of the pieces of the disc we cut, and the number 3 is a count of

the pieces we shaded," or "The number 4 is a count of the parts of the paper we got by folding, and the number 3 is a count of the parts that are colored."

The teacher will, of course, recognize the familiar fraction context. However, it is recommended that he continue to talk about pairs of whole numbers until his pupils have understood that the usual fraction numeral is nothing more than notation for a pair of numbers associated with counting collections of *pieces* of a unit.

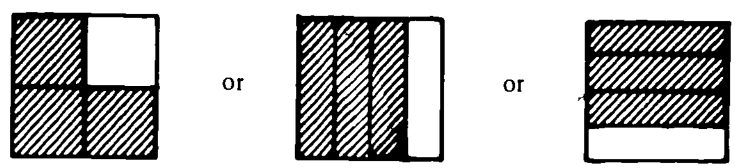
For pupils who have had little or no directed experience with ordered pairs of whole numbers in earlier grades, it may be necessary to repeat this kind of activity many times. Repeat with different models of units in different kinds of settings in which pupils assign ordered pairs of whole numbers to the models through (a) counting the number of pieces or parts or subregions of the same size into which a unit has been separated and (b) counting the number of pieces or parts or subregions which have been shaded or colored or given away or whatever.

The physical acts of cutting, separating and folding unit models should be done by the pupils and the teacher. In first experiences with writing ordered pairs of whole numbers to record the result of an experiment, it is suggested that teachers not use precut or pre-partitioned models; instead, let the physical act of partitioning a unit be part of the learning activity. Also, note that the word *divide* is not used to describe the physical act of partitioning a thing. The word *divide* is reserved for classroom use in talking about the mathematical operation of division on numbers.

obj. 8a

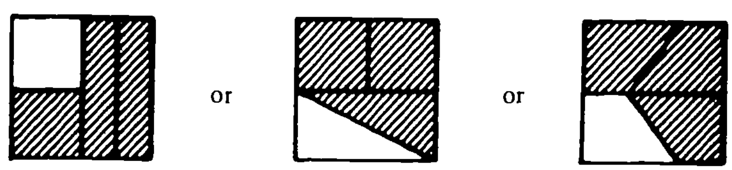
14. Once pupils have acquired the ability to assign number pairs to physical models, or pictures of models, in which units have been separated into parts or subregions which are congruent to each other, as illustrated in (a), they should be presented with models

(a)



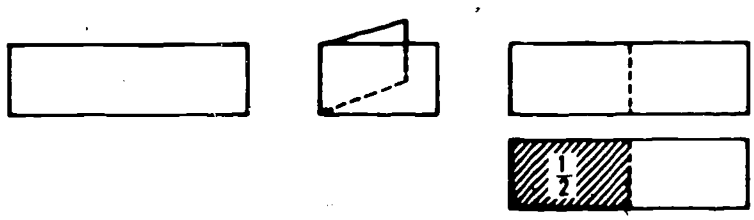
which have been partitioned into non-congruent parts, as illustrated in (b). Then ask, "What fraction can be assigned to the shaded parts in each of these models?" Have the pupils discuss and verify the assignment, $\frac{3}{4}$, to each model in the illustrations.

(b)

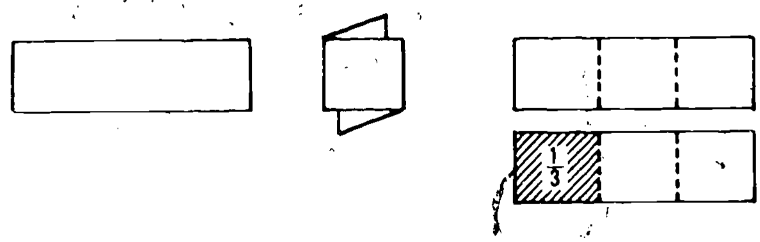


obj. 8a

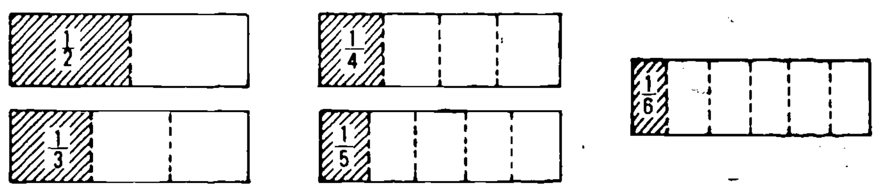
15. Prepare fraction makers for pupils by cutting out strips of paper which measure about 3 inches by 8 inches. (a) Give each pupil a strip and direct him to fold along a line so that the ends of the strip fit together. Then have him unfold the strip, shade one subregion and write the fraction name for the pair of whole numbers which describes the experiment.



(b) Give each pupil a second strip and direct him to fold the strip as in the illustration below. This activity will involve some trial and error. The teacher may demonstrate with a strip; however, let each student work with folding until he has accomplished the fitting together of the parts of the strip. Have him unfold the strip, shade one subregion and write the fraction name which describes the experiment.



(c) Continue having pupils fold strips and shade one of the resulting subregions until on each pupil's desk there is a set of strips, shaded and labeled, as pictured here.



(d) Ask, "What do you notice about the shaded parts of the strips?" Suppose one pupil responds, "This is bigger than that." Write the statement on the board and ask another student if he can show that *this* is bigger than *that*. The necessity of names for *this* and *that* will arise, and then you can cross out the words *this* and *that* and write

$\frac{1}{2}$ ~~This~~ is bigger than $\frac{1}{3}$ ~~this~~.

Out of such discussions can also arise statements such as, "If I shaded all of the pieces in each strip then I would have $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$, $\frac{5}{5}$ and $\frac{6}{6}$." Also, pupils may notice that two $\frac{1}{5}$'s and three $\frac{1}{5}$'s make five $\frac{1}{5}$'s, and the like. The teacher's continued encouragement to pupils to talk about what they see can provide the first intuitive recognition of relationships among fractions. The encouragement to use number names when talking about pieces will help to develop the language of fractions. However, one must be careful not to insist on mastery of facts about fractions until pupils possess the concept of fraction.

obj. 8a

16. Repeat activity 15 except let the strips be of different length or width than the strip which was provided for the previous activity. Although one does not explicitly point out to young learners that fraction numerals such as $\frac{2}{3}$ represent an abstract idea, the teacher needs to provide experiences with different models of units and with quantifying parts of those units. Thus, the pupil abstracts the idea of fraction without regard to any particular model.

obj. 9

17. Bring to class retail store advertisements which include symbols such as $\frac{3}{5}$, $\frac{3}{88}$, $\frac{5}{2}$, $\frac{2}{35}$, $\frac{3}{10}$ and the like, and discuss with pupils the physical models with which these ordered pairs of counting numbers are associated. For instance,

$\frac{3}{5}$ may mean 3 pieces of bubble gum for a nickel;
$\frac{3}{88}$ may mean 3 cans of peaches for 88 cents;
$\frac{5}{2}$ may mean 5 pairs of socks for 2 dollars;
$\frac{2}{35}$ may mean 2 boxes of facial tissues for 35 cents;
$\frac{3}{10}$ may mean 3 blouses for 10 dollars.

These symbols for ordered pairs of counting numbers name rates, not fractions. The context in which the symbol is used is the clue to its meaning. Ask pupils to find other examples of symbols for ordered number pairs which do not name fractions. They can bring in newspaper advertisements or they can draw pictures depicting their out-of-school experiences in which ordered pairs are used. Assist them to discriminate between number pairs which are fractions and number pairs which are not fractions. The distinction becomes increasingly important when the children study operations on fractions. For instance, if $\frac{3}{88}$ and $\frac{2}{88}$ are fractions, then one writes that $\frac{3}{88} + \frac{2}{88} = \frac{5}{88}$. However, if $\frac{3}{88}$ and $\frac{2}{88}$ are rate pairs, then $\frac{3}{88} + \frac{2}{88} = \frac{5}{176}$. That is if one buys three cans of one brand of peaches for 88 cents and two cans of another brand of peaches for 88 cents, the result is *not* 5 cans of peaches for 88 cents, but 5 cans for 176 cents.

obj. 8b, 9

18. Experiences in associating ordered pairs of whole numbers with partitioning sets should be developed carefully in order to avoid confusing rates and fractions. For instance if there are six children, four boys and two girls, in the first row, the set may be used to distinguish between a rate and a fraction.

The number pair (2,4) or $\frac{2}{4}$ which tells the ratio of the number of girls in the set to the number of boys in the set is a rate and is read *"two to four."* Whereas, the number pair (2,6) or $\frac{2}{6}$ which tells the ratio of the number of girls in the set to the number of students in the set is a fraction and is read *"two out of six."*

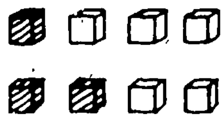
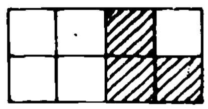
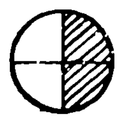
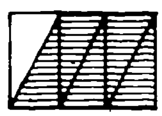
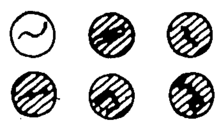
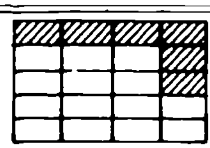
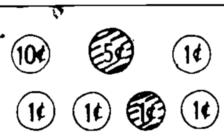
In the development of the fraction interpretation of number pairs associated with finite sets of objects, choose physical models of unit sets, and in discussions with pupils, ask questions similar to these which pertain to the unit set illustrated above.

(a) "How many members do you see in the set of children?" Record the number (6).
(b) "How many members (or girls) do you see in the subset of girls?" Record the number (2).
(c) "You now have two whole numbers. In what order do we write them? . . . (2,6) or $\frac{2}{6}$ . . . That is right. We write the count of members or pieces in the *unit set* after or under the count of members or pieces in the subset . . . We write the two numbers in this order because that's the way everybody else does." Pupils need to know that the way in which one records number ideas is man-invented and often quite arbitrary.

Although the teacher need not make explicit the relationships

(given) unit	$\longleftrightarrow$	(given) set,
pieces of unit	$\longleftrightarrow$	members of set,
pieces shaded or marked in some way	$\longleftrightarrow$	members specified by common property of subset,

it is helpful to learners if the teacher's language and classroom activities suggest the parallels as in the following chart.

Partition of Unit (Set)	No. of Pieces (or members) in Unit Set	No. of Pieces (or members) Shaded	Fraction Name
	8	3	$\frac{3}{8}$
	8	3	$\frac{3}{8}$
	4	2	$\frac{2}{4}$
	4	2	$\frac{2}{4}$
	6	5	$\frac{5}{6}$
	6	5	$\frac{5}{6}$
	20	6	$\frac{6}{20}$
	20	6	$\frac{6}{20}$

obj. 9

10. Provide wooden trays of balloons, bubble gum, pencils or other articles which one would find in a general store and which pupils would be likely to purchase. Prepare the usual rate pair signs for the trays, such as $\boxed{\frac{3}{10}}$ for balloons; $\boxed{\frac{1}{2}}$ for bubble gum; $\boxed{\frac{2}{5}}$ for pencils.

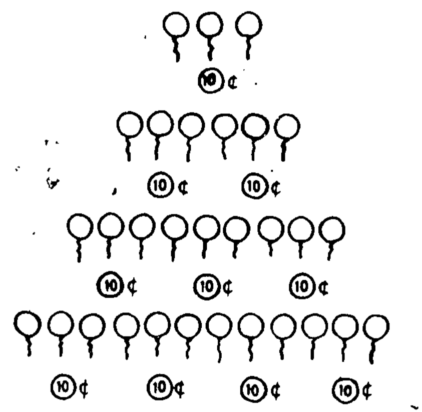
Pupils will have no trouble interpreting the signs, reading them "3 balloons for 10 cents," or "1 piece of gum for 2 cents" or "2 pencils for 5 cents." The discussion should lead to the generalization that order in writing each number pair is important; that is $\frac{3}{5}$ tells a different rate than does $\frac{5}{3}$. Then ask "What would be the ordered number pair which tells the rate of exchange for six balloons? for nine balloons? for 12 balloons?" If advisable, have pupils lay out the corresponding sets of balloons and coins as illustrated. Otherwise, have them draw the sets of balloons and coins.

(a) $\frac{3}{10}$ (read "3 for 10")

(b) $\frac{6}{20}$ (read "6 for 20")

(c) $\frac{9}{30}$ (read "9 for 30")

(d) $\frac{12}{40}$ (read "12 for 40")



After the class has discovered several number pairs involving the rate of exchange of balloons for pennies, ask "Why do you suppose that the store manager does not have a sign like this

$$\boxed{\frac{3}{10}, \frac{6}{20}, \frac{9}{30}, \frac{12}{40}, \frac{15}{50}}$$

over the tray of balloons?" Pupil responses and discussion should prompt generalizations that all of the above named pairs tell the *same* rate; that it is enough to advertise the basic rate of exchange; and that any purchaser of balloons could figure out the other number pairs. It is important that pupils understand

- that each number pair in the above set is associated with a different physical experience, that is, 3 balloons for 10 cents is not the same experience as 6 balloons for 10 cents, and hence one does not say that $\frac{3}{10}$ is equal to $\frac{6}{20}$; but
- that the number pairs in the set $\left\{\frac{3}{10}, \frac{6}{20}, \frac{9}{30}, \frac{12}{40}, \frac{15}{50} \ldots\right\}$ all represent exactly one rate. The rate pairs are said to be equivalent and, given the basic rate pair, all equivalent rate pairs can be determined. Since pupils probably have more real world experience with rates than they do with fractions, it is a good idea for teachers to make use of these experiences in order to develop the concept of a set of equivalent number pairs.

Other physical models for writing and generating sets of equivalent rate pairs include those employed in planning for a class party, such as

(1) three mints for one pupil, six mints for two pupils, nine mints for three pupils, . . ., denoted $\left\{\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \frac{15}{5}, \ldots\right\}$

(2) one lemon for 4 cups of lemonade, two lemons for 8 cups, three lemons for 12 cups . . ., denoted $\left\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \frac{5}{20}, \ldots\right\}$

Still other sets of rate-pairs are those used in solving measurement problems, such as

(3) miles per hour $\left\{\frac{55}{1}, \frac{110}{2}, \frac{165}{3}, \ldots\right\}$, read "55 miles in one hour, 110 in 2, 165 in 3, and so on."

(4) liquid ounces to cups $\left\{\frac{8}{1}, \frac{16}{2}, \frac{24}{3}, \ldots\right\}$ read "8 ounces to 1 cup, 16 ounces to 2 cups, and so on."

obj. 10

20. Working with paper strips as fraction makers (See Activities #15 and #16 above), have pupils partition the units by folding or cutting them; then have them shade some of the pieces and identify the associated number pairs and fraction symbols as indicated in the chart.

Partition of Unit	No. of Pieces in Unit	No. of Pieces Shaded	Fraction
	2	1	$\frac{1}{2}$
	4	2	$\frac{2}{4}$
	6	3	$\frac{3}{6}$
	8	4	$\frac{4}{8}$
	10	5	$\frac{5}{10}$
	12	6	$\frac{6}{12}$

Ask "What can you say about the shaded parts of the units (strips)?"
Record pupil responses on the board and discuss. Among responses may be one such as "The same amount of paper is shaded in each strip." If asked, "Is the fraction $\frac{1}{2}$ the same as the fraction $\frac{2}{4}$?", the response should be "No, because the partitions are not the same." Further discussion should lead to the agreement that although the units are not the same they are equivalent (same amount of paper) and although the partitions are not the same, the shaded portions of each unit are equivalent (same amount of paper). Thus, the number pairs or fractions are said to be equivalent, and one can say of the following sets that the fractions are equivalent to each other.

$$\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12} \ldots\right\}$$

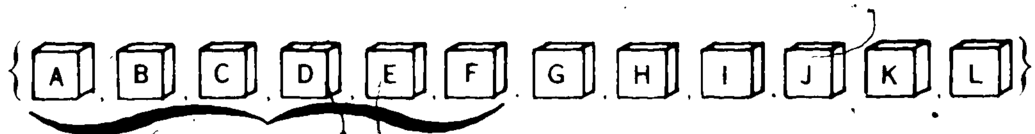
Ask, "Are there other number pairs or fractions which we could include in this set?" If necessary, continue partitioning units and shading parts until the pupils decide that there is no *last* fraction in a set of equivalent fractions. Thus, one writes the following.

$$\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \frac{7}{14}, \frac{8}{16} \ldots\right\}$$

obj. 10 21. Repeat the above activity except substitute a unit set of discrete objects for the unit strip of paper.

An egg carton filled with plastic eggs is a good model because it is easy to identify the subsets of the set of eggs by partitioning the carton.

If one used a unit set of 12 blocks, the physical partitionings and resulting data may be represented as follows.

(a) 

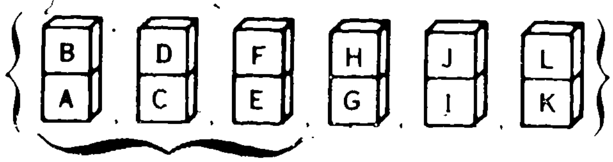
Subset to be given to Jack

The number of pieces (discrete objects) in unit set is 12.

The number of pieces in subset to be given to Jack is 6.

The fraction is $\frac{6}{12}$.

(b)



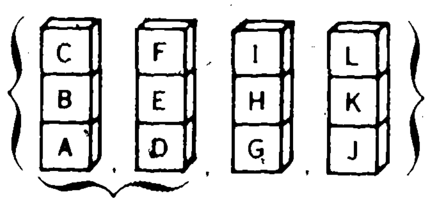
Subset to be given to Jack

The number of pieces in unit set (after partitioning the unit set into subsets or stacks of 2 each) is 6.

The number of pieces (stacks) in subset to be given to Jack is 3.

The fraction is $\frac{3}{6}$.

(c)



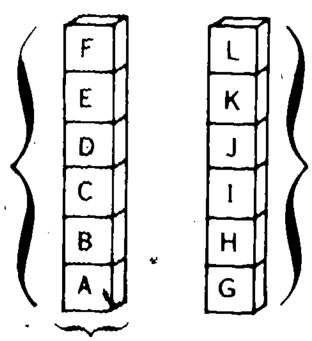
Subset to be given to Jack

The number of pieces in unit set (after partitioning unit set into subsets or stacks of 3 each) is 4.

The number of pieces (stacks) in subset to be given to Jack is 2.

The fraction is $\frac{2}{4}$.

(d)



Subset to be given to Jack

The number of pieces in unit set (after partitioning unit set into subsets or stacks of 6 each) is 2.

The number of pieces (stacks) in subset to be given to Jack is 1.

The fraction is $\frac{1}{2}$.

Although the partition in each of the above illustrations is not the same, the measures of the subsets given to Jack are the same. Thus, one says that the set of equivalent fractions $\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{6}{12}\right\}$ identifies the same number of elements in the unit set. Note that in this case it is inappropriate to include the number pairs $\frac{4}{8}$, $\frac{5}{10}$, $\frac{7}{14}$ or any other in the set.

obj. 10 22. Expand or build on activities above to provide practice in generating members of a set of equivalent fractions. Set up work sheets as follows.

In each of the following exercises, write ten other fractions that belong to the same set as the given fraction.

(a) $\{\frac{1}{2},$ $\}$

(b) $\{\frac{3}{4},$ $\}$

(c) $\{\frac{5}{3},$ $\}$

(d) $\{\frac{5}{8},$ $\}$

(e) $\{\frac{8}{6},$ $\}$

(f) $\{\frac{5}{5},$ $\}$

obj. 10,14a 23. After pupils have had much practice with quantifying physical models of various partitionings of units (and unit sets) and practice in generating sets of equivalent fractions, ask them "How did you generate the sets of equivalent fractions, given one fraction?"

Class discussion in search of a general pattern should lead to the discovery that, given a fraction in the form $\frac{a}{b}$, where a and b are whole numbers, $b \neq 0$, one can find other members of the set by multiplying each of the whole numbers in the given pair by the same number or dividing each of the whole numbers in the given pair by the same number (if division is possible).

For instance, given the fraction $\frac{2}{3}$ or the fraction $\frac{8}{12}$, one can generate other members of the set as

$$\left\{ \frac{2}{3}, \frac{2 \times 2}{3 \times 2} = \frac{4}{6}, \frac{2 \times 3}{3 \times 3} = \frac{6}{9}, \frac{2 \times 4}{3 \times 4} = \frac{8}{12}, \frac{2 \times 5}{3 \times 5} = \frac{10}{15} \cdots \frac{2 \times n}{3 \times n} \cdots \right\}$$

where n is a whole number, or

$$\left\{ \frac{8}{12}, \frac{8 \div 2}{12 \div 2} = \frac{4}{6}, \frac{8 \div 4}{12 \div 4} = \frac{2}{3}, \frac{2 \times 3}{3 \times 3}, \frac{2 \times 5}{3 \times 5}, \ldots \right\}.$$

In the latter set, the given fraction was not in basic form. Hence, dividing both whole numbers in the number pair by common factors, two more equivalent fractions were generated. Once the basic fraction is found, it is then an easy matter to generate all others.

obj. 11 24. Ask if the fractions $\frac{2}{3}$ and $\frac{8}{12}$ are equivalent fractions.

(a) Allow pupils to partition paper strips or other models of units, if they find it necessary.

$\frac{2}{3}$ $\frac{8}{12}$

Matching the shaded regions of equivalent units permits them to conclude that $\frac{2}{3}$ is equivalent to $\frac{8}{12}$

(b) Pupils working at a more abstract level may reason as follows.

Now $\frac{2}{3}$ is a basic fraction. Therefore, the set of fractions equivalent to $\frac{2}{3}$ is.

$$\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15} \dots\right\}.$$

Since $\frac{2}{3}$ and $\frac{8}{12}$ belong to the same set, they are equivalent.

(c) Another test for equivalence of two fractions is one which students may discover through a guided search for patterns. On the board write a set of equivalent fractions such as $\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \frac{12}{18}, \dots\right\}$ and ask, "What other pattern of relationship can you find for any two fractions in the set? Consider $\frac{4}{6}$ and $\frac{6}{9}$. Of course you can draw pictures of units and partition regions, but is there a quicker way? There is no whole number n such that $\frac{4 \times n}{6 \times n} = \frac{6}{9}$. Neither is there a whole number m such that $\frac{4}{6} = \frac{6 \div m}{9 \div m}$. Any other ideas?

Maybe it will help to look at the number pairs written in this form.

(4, 6) (6, 9) or $\frac{4}{6} \times \frac{6}{9}$

What can you say about the products, 4×9 and 6×6?"

Once this relationship on the number pairs is pointed out, have the class try other pairs of fractions from the set. Ask

"How about $\frac{2}{3}$ and $\frac{6}{9}$? Does $2 \times 9 = 3 \times 6$?" (Yes, since $18 = 18$)

"How about $\frac{4}{6}$ and $\frac{10}{15}$? Does $4 \times 15 = 6 \times 10$?" (Yes, since $60 = 60$)

"Try others. Generate a few more fractions which belong to the given set and try them."

Ask, "Do you suppose the same pattern holds for other sets? Try the set

$$\left\{\frac{6}{5}, \frac{12}{10}, \frac{18}{15}, \frac{24}{20}, \frac{30}{25} \dots\right\}$$

Consider $\frac{6}{5}$ and $\frac{18}{15}$. Does $6 \times 15 = 5 \times 18$?

Consider $\frac{12}{10}$ and $\frac{18}{25}$. Does $12 \times 25 = 10 \times 30$?"

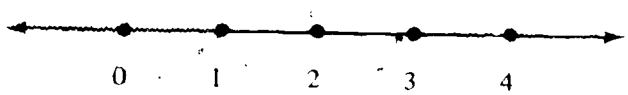
Then ask, "Can you use this relationship to find out if $\frac{9}{12}$ and $\frac{4}{6}$ are equivalent? Can you verify your response?"

The relationship which emerges from this kind of guided discovery is called the test for equivalence of ordered number pairs or fractions.

For all whole numbers p, q, r and s with $q \neq 0$ and $s \neq 0$, $\frac{p}{q}$ is equivalent to $\frac{r}{s}$ if and only if $p \times s = q \times r$.

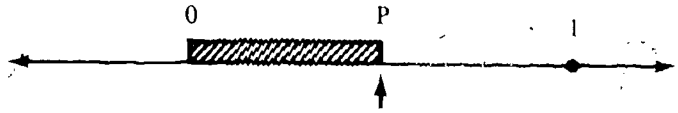
obj. 12J4a

25. Have pupils consider the *old* number line with which they have worked in ordering the whole numbers.



Recall that whole numbers have been associated with points on the line according to the agreement that the points are endpoints of unit segments and the whole numbers are measures or counts of unit segments from the point at zero.

The ensuing class discussion should follow this pattern. Partition the unit segment (between 0 and 1) into 2 pieces and shade 1 of the pieces such that 0 is one endpoint of the shaded piece.

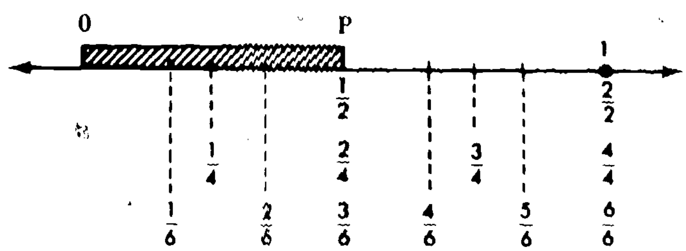


Ask, "With what number could we associate the other endpoint of the shaded piece?" Since it is the endpoint of 1 out of 2 pieces of the unit segment, let the ordered number pair (1, 2) or $\frac{1}{2}$ be associated with point P.

Partition the unit segment (between 0 and 1) into 4 pieces and shade the 2 pieces marked off successively from 0. Following the pattern already established, associate the number pair $\frac{1}{4}$ with the non-zero endpoint of the first shaded piece and the number pair $\frac{2}{4}$ with the non-zero endpoint of the segment composed of the two shaded pieces. Note that this point is also at P.

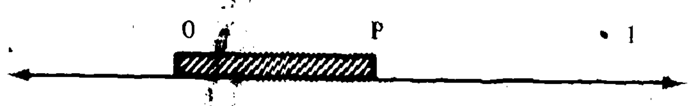
Partition the unit segment into 6 pieces and shade the first 3 pieces from 0. Following the pattern of associating number pairs with the endpoints of pieces of the unit segment, one finds that $\frac{3}{6}$ is associated with the point at P.

Partitioning the unit segment and associating number pairs with the endpoints of pieces results in the following figure.



At this point in the class discussion it is probably a good idea to ask, "What about the point at 0? What number pair describes the number of pieces marked off from zero?" When partitioning the unit segment into 2 pieces there were 0 shaded pieces from 0. Thus $\frac{0}{2}$ is the number pair assigned to that point; when partitioning the unit segment into 4 pieces, $\frac{0}{4}$ is the number pair assigned to that point; and so on.

Continuing activities in partitioning the unit segments into 8 pieces, 10 pieces or 12 pieces and shading 4 pieces, 5 pieces or 6 pieces, respectively, from the zero-point will result in this figure. (Some of the markings are omitted.)



$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$
$\frac{0}{4}$	$\frac{2}{4}$	$\frac{4}{4}$
$\frac{0}{6}$	$\frac{3}{6}$	$\frac{6}{6}$
$\frac{0}{8}$	$\frac{4}{8}$	$\frac{8}{8}$
$\frac{0}{10}$	$\frac{5}{10}$	$\frac{10}{10}$
$\frac{0}{12}$	$\frac{6}{12}$	$\frac{12}{12}$
•	•	•
•	•	•
•	•	•

Note that

(a) All of the equivalent fractions in the set $\left\{\frac{0}{2}, \frac{0}{4}, \frac{0}{6}, \frac{0}{8}, \frac{0}{10}, \frac{0}{12}, \ldots\right\}$ are associated with *one and only one* point, the same point with which the whole number 0 is associated.

(b) All of the equivalent fractions in the set $\left\{\frac{2}{2}, \frac{4}{4}, \frac{6}{6}, \frac{8}{8}, \frac{10}{10}, \frac{12}{12}, \ldots\right\}$ are associated with *exactly one* point, the same point with which the whole number 1 is associated.

(c) All of the equivalent fractions in the set $\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \ldots\right\}$ are associated with exactly *one* point, the point at P. One may ask, then, what is the number associated with point P? The question implies there is *exactly one* number corresponding to point P.

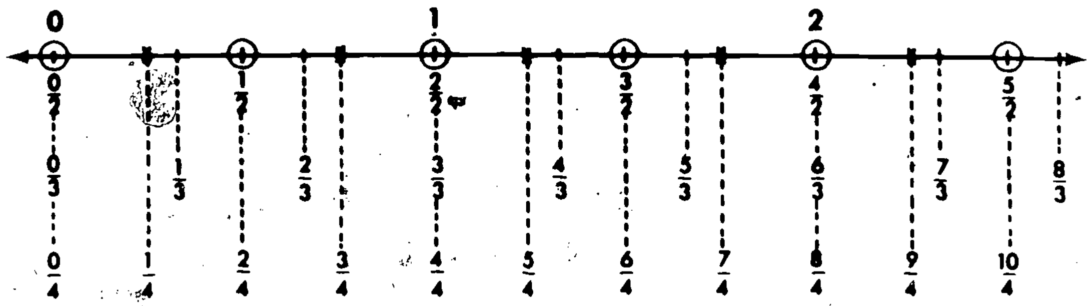
It seems reasonable to define the set $\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \ldots\right\}$ to be the unique number associated with the point P on this number line. In fact, the set is called *the rational number* associated with P and can be represented by any one of the numbers of the set. That is, the ordered number pair $\frac{3}{6}$ represents the same number as does the number pair $\frac{1}{2}$ or $\frac{5}{10}$. Pupils should not be penalized for writing $\frac{3}{6}$ or any other fraction in the set rather than $\frac{1}{2}$ as answers to exercises involving this particular rational number, unless they have been directed to write the solution in basic fraction form or in lowest terms. In that case, they should understand that they are expected to select an equivalent number pair in which the two numbers (that is, the numerator and denominator) are relatively prime to each other or, in other words, have no common factor other than 1 (one).

obj. 12 26. Other activities similar to the above should be provided. Pupils should have a number of experiences in which they

(a) partition unit segments on a number line and

(b) determine the infinite set of equivalent fractions associated with each of the points determined by the partitions.

Example



Pupil's attention should be called to those particular infinite sets that are associated with the same points as are the whole numbers. Then one can say the following.

$$0 = \left\{ \frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \frac{0}{4}, \frac{0}{5}, \ldots \right\}$$

$$1 = \left\{ \frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \ldots \right\}$$

$$2 = \left\{ \frac{2}{1}, \frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \frac{10}{5}, \ldots \right\}$$

$$3 = \left\{ \frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \frac{15}{5}, \ldots \right\}$$

In general, for all whole numbers *a*,

$$a = \left\{ \frac{a}{1}, \frac{2 \times A}{2}, \frac{3 \times A}{3}, \frac{4 \times A}{4}, \frac{5 \times A}{5}, \ldots \right\}$$

and one may select any member of the set to represent the whole number.

obj. 13

27. Activities involving decimal fractions might begin with examination of sets of equivalent fractions such as those below and the question, "What sets contain fractions in which the denominator is 10 or 100 or 1000 or some other power of 10?"

$$A = \left\{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \frac{4}{24}, \frac{5}{30}, \ldots \right\}$$

$$B = \left\{ \frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \frac{4}{20}, \frac{5}{25}, \frac{6}{30}, \ldots \right\}$$

$$C = \left\{ \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \frac{5}{15}, \frac{6}{18}, \ldots \right\}$$

27

$$D = \left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \ldots\right\}$$

$$E = \left\{\frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \frac{15}{25}, \frac{18}{30}, \ldots\right\}$$

$$F = \left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \frac{12}{18}, \ldots\right\}$$

$$G = \left\{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \frac{15}{20}, \frac{18}{24}, \ldots\right\}$$

$$H = \left\{\frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \frac{25}{30}, \frac{30}{36}, \ldots\right\}$$

$$I = \left\{\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \frac{6}{6}, \ldots\right\}$$

$$J = \left\{\frac{4}{3}, \frac{8}{6}, \frac{12}{9}, \frac{16}{12}, \frac{20}{15}, \frac{24}{18}, \ldots\right\}$$

Some pupils may need to generate more fractions in each of the sets in order to determine the following.

In A, there are no decimal fractions.

In B, there are $\frac{2}{10}, \frac{20}{100}, \frac{200}{1000}, \ldots$

In C, there are no decimal fractions.

In D, there are $\frac{5}{10}, \frac{50}{100}, \frac{500}{1000}, \ldots$

In E, there are $\frac{6}{10}, \frac{60}{100}, \frac{600}{1000}, \ldots$

In F, there are none.

In G, there are $\frac{75}{100}, \frac{750}{1000}, \ldots$

In H, there are none.

In I, there are $\frac{1}{1}, \frac{10}{10}, \frac{100}{100}, \ldots$

In J, there are none.

Pupils should be told that fractions in which the denominators are 1 or 10 or 100 or 1000 or a power of ten (that is, 10^n where n is a counting number) are called *decimal fractions*.

After pupils have found the sets which contain decimal fractions, they should be asked to try to find a rule or test for deciding whether or not any given fraction is equivalent to a decimal fraction. They should notice that for some fractions in basic fraction form, for instance, $\frac{1}{5}$, $\frac{1}{2}$, $\frac{3}{5}$, $\frac{3}{4}$, $\frac{1}{1}$, there are equivalent decimal fractions, while for other fractions such as $\frac{1}{6}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{5}{6}$, $\frac{4}{3}$, there are none. (Note that basic fractions having denominators of 1, 2, or 5 or powers of 2 or 5 or products of powers of 2 and 5 are the only fractions for which there are equivalent decimal fractions. Pupils should be encouraged to discover this test for themselves.)

obj. 14

28. To introduce alternate notations for fractions other than the ratio-like form $\frac{a}{b}$ (read "*a* over *b*"), pupils could be asked to find a test for deciding whether or not a given fraction such as $\frac{3}{2}$ belongs to one of the following categories.

(a) Fractions less than 1.

(b) Fractions greater than 1 and less than 2.

(c) Fractions greater than 2 and less than 3.

Pupils' prior work with segments of the number line should help. For instance, in examining number lines such as those in activities 25 and 26, pupils will probably describe the segment whose end point is associated with $\frac{3}{2}$ with phrases such as—

More than 1 and less than 2 unit segments;

$\frac{1}{2}$ of a unit segment more than 1 unit segment;

1 and $\frac{1}{2}$ more; or

1 plus $\frac{1}{2}$.

It will then seem reasonable to introduce the idea that another way of expressing the fraction $\frac{3}{2}$ is to say or write "$1 + \frac{1}{2}$" or, using the conventional shorthand way of writing that relationship, $\frac{3}{2} = 1\frac{1}{2}$. The numeral "$1\frac{1}{2}$" is called a *mixed numeral* and is just another way of naming the fraction $\frac{3}{2}$.

obj. 14

29. Still another numeral form for fractions makes use of the place value scheme for writing numerals. The following activity provides readiness for writing fractions in decimal notation.

Write the following sequences of numbers and related questions on the board. (Note that the three dots ". . ." indicate an unending sequence in the direction of the three dots.)

(a) 1, 10, 100, 1000, 10,000, . . .
(b) . . . 10,000, 1000, 100, 10, 1.
(c) $10^0, 10^1, 10^2, 10^3, 10^4, 10^5, \ldots$
(d) $\ldots 10^5, 10^4, 10^3, 10^2, 10^1, 10^0$.

(1) In which sequences are the numbers increasing? (a and c)
(2) In which sequences are the numbers decreasing? (b and d)
(3) What are some of the other numbers in the sequences?
(4) What is the pattern of increase from left to right in sequences a and c? (Multiply each term by 10 to get the next term.)
(5) What is the pattern of increase from right to left in sequences d and b? (Multiply each term by 10 to get the next term.)
(6) What is the pattern of decrease from left to right in sequences b and d? (Multiply each term by $\frac{1}{10}$ to get the next term.)
(7) Where else have you seen and used sequences b or d? (See below) Pupils may more easily recognize the sequences d and b if the teacher draws in short vertical lines as illustrated here.

. . .	10,000	1000	100	10	1
. . .	10^4	10^3	10^2	10^1	10^0

The sequences are, of course, the same as those found in the headings of place value charts. After pupils are skillful in generating members in the above sequences, then they should consider the following sequences.

$\ldots, \quad 10{,}000, \quad 1000, \quad 100, \quad 10, \quad 1, \ldots$

$\ldots, \quad 10^4, \quad 10^3, \quad 10^2, \quad 10^1, \; 10^0, \ldots$

Ask the questions, "How do these sequences differ from b and d?" (The three dots indicate the sequences are non-ending in both directions.) "Can you find other numbers which belong in the sequences to the right?"

Answers are as follows.

$$\ldots, \quad 10{,}000, \quad 1000, \quad 100, \quad 10, \quad 1, \quad \frac{1}{10}, \quad \frac{1}{100}, \quad \frac{1}{1000}, \ldots$$

$$\ldots, \quad 10^4, \quad 10^3, \; 10^2, \; 10^1, \; 10^0, \; 10^{-1}, \quad 10^{-2}, \quad 10^{-3}, \ldots \text{ or}$$

$$10^4, \quad 10^3, \; 10^2, \; 10, \quad 1, \; \frac{1}{10}, \quad \frac{1}{10^2}, \quad \frac{1}{10^3}, \ldots$$

(The teacher should not expect all pupils to be able to use all of the various notations for integral powers of ten. Teachers will know which notations are most appropriate for their pupils.)

The next step would be to extend the place value chart just as the above sequences were extended. If pupils are secure in their knowledge of the place value scheme of writing numerals it could be challenging and fun to play around with extending the chart and deciding on some rule for reading and writing numerals to represent, for instance, the following count of places in an extended place value chart.

...	1000	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	...
	3	1	2	6	5	4	

Reading the count of places is easy – "3 thousands 1 hundred 2 tens 6 ones 5 tenths 4 hundredths." However, in writing the numeral without the place-value labels of the chart, the pupils would write "3 1 2 6 5 4," which is 312 thousand and some, not the number in the chart. The teacher should ask questions which help pupils recall that in reading and writing numerals they have generally looked first at the digit in the one's place. That is, the one's place is the *reference place* to which one looks in deciding how to read or to write numerals. Thus, some sign is needed to mark the one's place. Perhaps the pupils can invent some sign, say ⧗, and write 3126 ⧗ 54 (read "3 thousand 1 hundred twenty-six and 5 tenths 4 hundredths.")

The teacher may leave the problem of reading "5 tenths 4 hundredths" in a more conventional way and of telling about the conventional mark for the reference place until after the following activity.

30. The teacher could say, "Let's consider the fraction $\frac{13}{10}$. What is another way to write or name this number?" If the response is "(13, 10)" or "$\frac{26}{20}$" or "$\frac{130}{100}$" or "any member from the set $\left\{\frac{13}{10}, \frac{26}{20}, \frac{39}{30}, \ldots\right\}$," the teacher should acknowledge the correctness of the response and continue to repeat the question. Someone will recall the mixed numeral form "1 $\frac{3}{10}$." Questions may be asked to elicit the responses "1 $\frac{6}{20}$" or "1 $\frac{30}{100}$", also. The question should be repeated for the fraction $\frac{3}{2}$. Correct responses would include $\frac{6}{4}$, $\frac{9}{6}$, $\frac{15}{10}$, $1\frac{1}{2}$, $1\frac{3}{6}$, $1\frac{5}{10}$, and the like. After repeating the question for other fractions, such as $\frac{9}{4}$, $\frac{36}{10}$ and $\frac{68}{5}$, the teacher could ask, "Do any of these numerals name decimal fractions? How about the mixed numerals? Do any of them name a decimal fraction?" Correct responses include the following.

For $\frac{13}{10}$: $\frac{13}{10}, \frac{130}{100}, \ldots, 1\frac{3}{10}, 1\frac{30}{100}, \ldots$

For $\frac{3}{2}$: $\frac{15}{10}, \frac{150}{100}, \ldots, 1\frac{5}{10}, 1\frac{50}{100}, \ldots$

For $\frac{9}{4}$: $\frac{225}{100}, \ldots, 2\frac{25}{100}, \ldots$

For $\frac{36}{10}$: $\frac{36}{10}, \frac{360}{100}, \ldots, 3\frac{6}{10}, \ldots$

For $\frac{68}{5}$: $\frac{136}{10}, \ldots, 13\frac{6}{10}, \ldots$

Conversion from fraction form to mixed numeral form depends on recognizing that any fraction may be written as the sum of two other fractions. For fractions greater than 1 (one), the pupil should find two addends such that one of the addends is less than 1 and the other addend is equivalent to a fraction of the form $\frac{a}{1}$, where a is a whole number. Then, according to the agreement in activity 26, one may write the whole number a for the fraction $\frac{a}{1}$. For instance,

$$\frac{13}{10} = \frac{10}{10} + \frac{3}{10} = \frac{1}{1} + \frac{3}{10} = 1 + \frac{3}{10} = 1\frac{3}{10}.$$

$$\frac{9}{4} = \frac{8}{4} + \frac{1}{4} = \frac{2}{1} + \frac{1}{4} = 2 + \frac{1}{4} = 2\frac{1}{4}$$

$$\frac{225}{100} = \frac{200}{100} + \frac{25}{100} = \frac{2}{1} + \frac{25}{100} = 2 + \frac{25}{100} = 2\frac{25}{100}$$

It is an important skill to be able to write a given fraction as the sum of two other fractions. The only other skill a pupil needs to convert a fraction numeral to a mixed numeral is a skill in generating equivalent fractions. Until a pupil has studied the operation of division on fractions there is no mathematically sound rationale for interpreting the number pair $\frac{9}{4}$ ("9 over 4") as $9 \div 4$, or the number pair $\frac{36}{10}$ as $36 \div 10$.

After pupils are skillful in (a) identifying those fractions equivalent to decimal fractions, (b) determining the equivalent decimal fraction and (c) converting decimal fraction numerals to mixed numeral form, it is appropriate to introduce a place value chart such as

. . .	1000	100	10	1	. . .

and ask, "Suppose we continue the pattern to the right in the place value chart. What would the place readings be?" Some will recall from the readiness activity that one could extend the place value scheme this way.

...	1000	100	10	1 ⧗	$\frac{1}{10}$	$\frac{1}{100}$	...

Then ask, "How about using the new place value chart to write a place value numeral instead of the mixed numeral $13\frac{6}{10}$?" Pupils should respond with

...	10	1 ⧗	$\frac{1}{10}$	...
	1	3	6	

or 13 ⧗ 6

(read "thirteen and 6 tenths).

Continued use of the new place value chart to convert mixed numerals to place value numerals should not only reinforce the important idea of place value but also develop insight into the use of place value representation of fractions. Such place value representations are commonly called *decimal representations* or *decimals*. Of course, teachers will explain that in the United States one uses a dot called the *decimal point* to mark the reference place, that is, the one's place. Teachers should ask their pupils to find out what symbols other nations (such as the English or the French) use to mark the reference place in writing numerals.

obj. 15

31. To introduce place-value or decimal representations of non-decimal fractions, a teacher might ask his pupils how they think the fraction numeral $\frac{5}{3}$ or the equivalent mixed numeral $1\frac{2}{3}$ could be written using the new place value chart.

(a) If pupils know how to generate the set of fractions equivalent to $\frac{5}{3}$ in order to find an equivalent decimal fraction they will start this way.

$$\left\{ \frac{5}{3}, \frac{10}{6}, \frac{15}{9}, \frac{20}{12}, \frac{25}{15}, \frac{30}{18}, \frac{35}{21}, \ldots \right\}$$

(b) The more able and sophisticated student will probably use the test for equivalent fractions in this way.

Suppose there is a whole number N so that $\frac{5}{3} = \frac{N}{10}$.

Then $5 \times 10 = 3 \times N$. However, there is no whole number N such that $50 = 3 \times N$. Thus, there is no equivalent decimal fraction with a denominator of 10. Then he may try letting $\frac{5}{3} = \frac{N}{100}$. Then $5 \times 100 = 3 \times N$. But there is no whole number N such that $500 = 3 \times N$. Thus, there is no equivalent decimal fraction with denominator of 100.

(c) Other pupils may reason this way. "If $\frac{5}{3}$ is equivalent to $\frac{N}{1000}$, then there is some number that I can multiply 3 by so that the product is 1000, and then I can multiply 5 by the same number and that product will be N. That is $\frac{5 \times \square}{3 \times \square} = \frac{N}{1000}$. What number goes in $\square$?"

(There is no whole number that multiplies by 3 to give the product 1000.)

Continuing in any of the methods available to them for trying to find a decimal fraction equivalent to $\frac{5}{3}$ should convince them that neither $\frac{5}{3}$ nor $1\frac{2}{3}$ can be written in decimal fraction form.

At this point in the study of decimals, the teacher can use the pupil's experiences with linear measurement and his knowledge of the practical or applied use of decimal representations for certain measures. For instance, in applications to the real world one often finds it necessary to record measurements in decimal form. For instance, one may wish to write $1\frac{2}{3}$ *yards* in decimal notation. In that case, one settles for an approximation to the actual measure. In the case of fractions $\frac{5}{3}$, $\frac{1}{6}$, $\frac{7}{12}$, $\frac{1}{3}$, and all other non-decimal fractions, it is suggested that the question of how to represent these fractions as decimals be approached through a method of successive approximations, as illustrated here. Consider fraction $\frac{1}{3}$.

Now

$$0.3 = \frac{3}{10} \quad \text{and } \frac{1}{3} > \frac{3}{10}.$$

$$0.33 = \frac{33}{100} \quad \text{and } \frac{1}{3} > \frac{33}{100} > \frac{3}{10}.$$

$$0.333 = \frac{333}{1000} \quad \text{and } \frac{1}{3} > \frac{333}{1000} > \frac{33}{100} > \frac{3}{10}.$$

$$0.3333 = \frac{3333}{10000} \quad \text{and } \frac{1}{3} > \frac{3333}{10000} > \frac{333}{1000} > \frac{33}{100} > \frac{3}{10}.$$

The decimal fractions $\frac{3}{10}, \frac{33}{100}, \frac{333}{1000}, \frac{3333}{10000}, \ldots,$ get closer and closer to $\frac{1}{3}$, since

$$\frac{1}{3} > \frac{3}{10} \text{ and } \frac{1}{3} - \frac{3}{10} = \frac{10-9}{30} = \frac{1}{30}$$

$$\frac{1}{3} > \frac{33}{100} \text{ and } \frac{1}{3} - \frac{33}{100} = \frac{100-99}{300} = \frac{1}{300}.$$

$$\frac{1}{3} > \frac{333}{1000} \text{ and } \frac{1}{3} - \frac{333}{1000} = \frac{1000-999}{3000} = \frac{1}{3000}.$$

$$\frac{1}{3} > \frac{3333}{10000} \text{ and } \frac{1}{3} - \frac{3333}{10000} = \frac{10000-9999}{30000} = \frac{1}{30000}.$$

The differences grow smaller, and although one cannot write as true statements about number that $\frac{1}{3} = \frac{3}{10}$ or $\frac{1}{3} = \frac{33}{100}$ or $\frac{1}{3} = \frac{333}{1000}$, one agrees that $\frac{1}{3}$ can be approximated by $\frac{3}{10}$ or 0.3; that an even better approximation to $\frac{1}{3}$ is $\frac{33}{100}$ or 0.33; and successive approximations to $\frac{1}{3}$ differ from $\frac{1}{3}$ by less and less. In applications to the real world rational approximations to $\frac{1}{3}$ are selected according to the degree of error or *goodness* of the approximation one is willing to accept.

32. After pupils have studied division of fractions and can justify the algorithm $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c}$, where *a, b, c* and *d* represent non-zero counting numbers, then one can argue as follows.

For all non-zero counting numbers p and q, $p \div q$ can be written in rational number notation as $\frac{p}{1} \div \frac{q}{1}$ and, using the division algorithm, $\frac{p}{1} \div \frac{q}{1} = \frac{p}{1} \times \frac{1}{q} = \frac{p}{q}$, thus, the fraction (or rational number) $\frac{p}{q}$ can be interpreted as p divided by q. (See the strand on Operations, Their Properties and Number Theory.)

Once pupils have acquired the quotient concept of rational numbers, then it is possible for them to determine (a) non-terminating decimal notation for all rational numbers, including the non-decimal fractional numbers, and (b) terminating decimal approximations to the nearest specified place. See the chart for instances of non-terminating decimals and decimal approximations for selected rational numbers.

Rational Numbers	Non-Terminating Decimal Notations	Decimal Approximations		
		To Nearest Tenth	To Nearest Hundredth	To Nearest Thousandth
$\frac{1}{3}$	*0.3333 . . .	0.3	0.33	0.333
$\frac{2}{7}$	*$0.285714\overline{285714}$ . . .	0.3	0.29	0.286
$\frac{1}{8}$	*$0.125\overline{000}$ . . .	0.1	0.13	*0.125
$\frac{1}{6}$	*$0.166\overline{66}$ . . .	0.2	0.17	0.167
$\frac{7}{12}$	*$0.583\overline{33}$ . . .	0.6	0.58	0.583
$\frac{5}{3}$	*$1.666\overline{66}$ . . .	1.7	1.67	1.667
$\frac{253}{100}$	*$2.530\overline{00}$ . . .	2.5	*2.53	*2.530

*Only in these cases does one say that the given rational number is equal to the rational number in decimal notation.

In other cases one says that the given rational number is approximately the same as the rational number in decimal notation.

Examples

$\frac{1}{6} = 0.166\overline{6}$ . . . $\qquad$ $\frac{1}{6}$ is approximately the same as 0.17 (to the nearest hundredth)

$\frac{1}{8} = 0.125\overline{00}$ . . . or $\qquad$ $\frac{1}{8}$ is approximately the same as 0.1 (to the nearest tenth)

$\frac{1}{8} = 0.125$

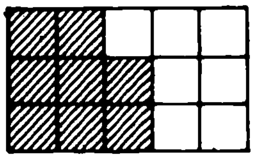
obj. 14,18

33. The teacher should occasionally check-up on his pupils' awareness of the many interpretations of ordered pairs of whole numbers. He may use as simple an activity as writing $\frac{8}{15}$ on the chalkboard and saying, "What do you think that means? Make a sketch, write a sentence or demonstrate in some way at least one meaning for the symbol." Possible responses are as follows.

(a) 8 crayons cost 15 cents (rate-pair interpretation).

(b) 8/15 is another way of writing the date, August 15 (ordinal pair interpretation).

(c) $\frac{8}{15}$ is associated with this partition of a rectangular disc (fraction interpretation).



(d) 8 boys out of the 15 boys in our class are in the band (ratio interpretation).

(e) $\frac{8}{15}$ is the quotient of 8 and 15, or $8 \div 15$ (quotient interpretation).

After the class examines the many interpretations proposed, the teacher should ask, "In which of the contexts could the number pair be considered a rational number?" (Answer: The fraction, ratio and quotient interpretations.)

The teacher should also check-up occasionally on the understanding that there are several different numerals which represent the same rational number. For instance, the numerals $\frac{17}{5}$, $3\frac{2}{5}$ and 3.4 all represent the same number, as does the infinite set $\left\{\frac{17}{5}, \frac{34}{10}, \frac{51}{15}, \ldots\right\}$ The choice one makes with regard to the numeral form one uses to express a fraction or rational number generally depends on the use or application one wishes to make of the number.

Integers

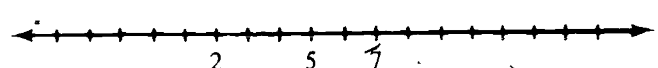
obj. 18 — 34. Have the pupils record the results of placing a thermometer in liquids which will cause the thermometer to fluctuate. Be sure that it is placed in a liquid which will cause the temperature reading to drop to below $0^{\circ}C$. Such a liquid would be a solution of water and alcohol which has been refrigerated and cooled below $0^{\circ}C$. Encourage the pupils to discuss the measures of temperatures lower than $0^{\circ}C$.

obj. 18,19 — 35. Another activity enabling pupils to have experiences which involve positive and negative integers is the postman game. The game involves pupils being homeowners and one pupil acting as postman. The postman delivers checks and bills. Each homeowner keeps track of income and outgo by treating the amounts on checks as positive numbers and amounts on bills as negative numbers. The game can be expanded into the study of operations on integers as the postman becomes confused and delivers the wrong mail to several people. (The teacher can find descriptions of this game in references listed in the bibliography.)

obj. 18 — 36. The elevator game appeals to many pupils as a way of thinking about positive and negative integers. A building has 4 basement floors and 10 floors above street level. The pupils take turns telling the elevator operator where they want to go by naming a positive or a negative integer. The game can also be extended to involve operations on integers if each pupil tells the elevator operator 2 numbers and notices his position relative to the street floor after combining the two trips.

obj. 19 — 37. A game could be played with all pupils standing on a stair landing. Each pupil draws a slip of paper telling him to go up 5 steps, down 9 steps, up 4 steps, down 4 steps and the like. In this activity the teacher should ask questions to develop the idea of opposites. The pupil should see that since the $^{+}4$ (up 4 steps) followed by $^{-}4$ (down 4 steps) leaves him in his original position, that $^{+}4$ and $^{-}4$ are opposites, that is, that $^{-}4$ is the opposite of $^{+}4$ and $^{+}4$ is the opposite of $^{-}4$.

obj. 19,20 — 38. If a pupil has had many experiences with the number line he could be asked to consider a number line that looks like this.



35

Ask him to fill in other numbers for the indicated points on the line so that he has a chance to suggest $^-1$, $^-2$, and $^-3$. The number line provides an excellent opportunity for developing the concept of opposites. Given a number line with unit segments to the right corresponding to steps to the right, and unit segments to the left corresponding to steps to the left, the teacher should ask questions such as the following.

(a) What is the opposite of 2 steps to the right? The opposite of $^+2$ is ________.

(b) What is the opposite of 5 steps to the left? Then the opposite of $^-5$ is ________.

(c) What is the opposite of 0 steps to the right? The opposite of 0 is ________. (Help the pupils see that zero is different from other numbers in that it is its own opposite.)

(d) What is the set of opposites of the whole numbers? The student may write this as $\{\ldots, ^-4, ^-3, ^-2, ^-2, 0\}$

(e) What is the set of numbers that are the opposites of the opposites of whole numbers? $\{0, ^+1, ^+2, ^+3, ^+4, \ldots\}$

Tell the pupil that the union of these two sets is the set of integers $\{\ldots, ^-3, ^-2, ^-1, 0, ^+1, ^+2\}$

obj. 18 39. Pupils themselves can suggest many activities involving the recording of gain and loss, such as the number of yards rushing in football games, net profit or net loss in business and similar examples.

obj. 18 40. Have the pupils locate places on the map that are above and below sea level. They can describe the locations using integers.

obj. 18 41. An activity that will appeal to older pupils is the use of a model from science in which the drawings represent an empty field, a bucket containing positive particles and a bucket containing negative particles. As the pupils place positive and negative charges into the field they will observe the results of combining particles of opposite charges. This gives the teacher an opportunity to emphasize the word *opposite* as it relates to the integers. The concept of neutralization adds much to this activity. The pupil may draw a circle around the neutralized particles ($\pm$) and this gives insight into combining or adding integers. An activity of this type can be found in the strand Operations, Their Properties and Number Theory.

obj. 20 42. As the pupils study the set of integers have them discuss various subsets such as the following.

All integers n such that

(a) $n > 3$	(f) $n > 0$
(b) $n < 2$	(g) $n \geq 0$
(c) $n > ^-2$	(h) $n < 0$
(d) $n > ^-3$ and $n < ^+5$	(i) $n \leq 0$
(e) $n > ^-3$ or $n < ^+5$	(j) $n > ^-1$ and $n < ^+1$

Tell the pupils that the set described in (f) is the set of positive integers; the set described in (g) is the set of non-negative integers; the set in (h) is the set of negative integers; the set in (i) is the set of non-positive integers; and the set described in (j) is the set whose only member is zero.

Give the pupils various subsets of the integers, and ask the pupils to place the integers in each set in order from least to greatest.

Examples

$\{^+3, ^-3, ^+1\}$ is $\{^-3, ^+1, ^+3\}$

$\{0, ^-2, ^+5, ^-3\}$ is $\{^-3, ^-2, 0, ^+5\}$

$\{^-3, ^-5, ^-2\}$ is $\{^-5, ^-3, ^-2\}$

SETS, NUMBERS AND NUMERATION

OBJECTIVES

The pupil should be able to do the following.

1. Tabulate and describe sets
2. Pick out from a given set, subsets having a specified common property
3. Identify common properties of a given set
4. Use the language of sets to describe and organize information
5. Read and write large numbers using period numeration
6. Translate large numbers into expanded exponential form
7. Demonstrate place value by using another system of numeration
8. Name the ordered pair of whole numbers associated with fractional parts of (a) units, (b) sets
9. Discriminate between an ordered pair of whole numbers used in a rate context and an ordered pair of whole numbers used in a fraction context
10. Generate a finite number of members of the set of equivalent fractions to which a given fraction belongs
11. Determine if two ordered number pairs are equivalent to each other
    (a) by inspection of sets of equivalent number pairs
    (b) by using the test for equivalence– if a, b, c and d are whole numbers, $b \neq 0, d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $a \times d = b \times c$
12. Identify the set of equivalent fractions associated with a given point on a number line
13. Tell whether or not a given fraction is equivalent to a decimal fraction
14. Record fractions
    (a) in basic fraction form
    (b) in mixed numeral form
    (c) in decimal notation
15. Find rational approximations in decimal notation, to the nearest tenth, to the nearest hundredth and to the nearest thousandth for given rational numbers using
    (a) the process of successive approximations
    (b) the process of long division
16. Find non-terminating decimal representation of given rational numbers
17. Discriminate among the interpretations of rational numbers as used in the context of fractions, of ratios and of quotients of whole numbers.
18. Identify and describe everyday situations which exemplify or require the use of directed numbers
19. Construct the set of opposites of the whole numbers and the opposites of the opposites which together form the set of integers
20. Order any two or more given integers

OPERATIONS, THEIR PROPERTIES AND NUMBER THEORY

INTRODUCTION

The purpose of this strand is two-fold. One is to build the concept of operations and their properties, and the other is to develop interest in number relationships through number theory.

In this guide the operations and their properties are separated from the algorisms or computation. The distinction between operations and their properties and computations is an important one. An operation is a particular association of a certain member of a set to a given pair of numbers of the set. Computation is the manipulation of numerals to determine the number that results from combining two numbers by means of an operation.

Operations and their properties in the guide are studied in terms of their meanings. The child is introduced to each of the four fundamental operations of arithmetic through some physical situation. The initial interpretation of the operation is derived from the physical situation.

The concept of number operations evolves from two main physical sources – one is the number associated with sets of discrete objects, and the other is the measurement of continuous quantities. Therefore, the activities in this strand are concerned with sets (discrete) and the number line (continuous).

In using the operations, the child must know which is applicable to the situation in the problem at hand. For example, the number pair 6, 2 can be associated with 4, with 8, with 3, or with 12. The child must select the appropriate operation for solution of his problem situation, and he must know which number is associated with the operation.

The four fundamental operations with integers or rationals cannot always be introduced to the pupil using physical situations. Therefore, separate activities are needed to introduce the child to an interpretation of these operations for these sets of numbers. As with whole numbers, writing symbols for the operations is more effectively understood by the child after generalizations have been firmly grasped. For example, to be able to write $^{+}3 + {}^{-}5 = {}^{-}2$ with understanding the child must have experience with interpreting a physical situation such as – if one goes east three miles ($^{+}3$) and then west five miles ($^{-}5$), his location is then two miles west of the place he started ($^{-}2$).

After acquiring a basic understanding of operations in a number system, the pupil may use this knowledge to explore number ideas through number theory. In working with operations one begins with a pair of numbers to which a single number is assigned by a specific operation; in studying number theory one encounters such experiences as looking *inside* a single number and studying the relationship between numbers of a particular set. For example, one may look closely at the single number 49 to find answers to questions as – Is it a prime number? Is it an odd number? Is it a square number? The child may also investigate number patterns in order to recognize numerous relationships of numbers – for example, extending patterns, skip counting, classifying numbers as odd or even, prime or composite, and many other topics included in typical modern elementary mathematics textbooks.

Investigating numbers and number patterns provides more challenging and appealing activities for a child to use in learning mathematical concepts and basic facts than the traditional drill activities or practice exercises. The study of number theory is especially interesting in that a solution to one problem very often becomes the basis for another problem.

In modern textbooks number theory is treated as a separate topic. In others the concepts are included under topics such as multiplication of whole numbers. It is important for the teacher to see that concepts of number theory should be taught as a foundation for other concepts. For example, the study of least common multiple would be necessary before addition of certain rational numbers.

In the early grades the child should learn about properties of operations by manipulating objects and observing the number relationships on which the properties are based. It is not as impoftant for him to know the names of the properties as it is for him to apply them when appropriate. In the upper grades the pupil should be able to identify the properties by name.

OPERATIONS, THEIR PROPERTIES AND NUMBER THEORY

Objectives Keyed to Activities	ACTIVITIES
	Number Theory
obj. 3, 12a	1. The teacher may mimeograph or make a transparency of the table below. Say to the pupils and point to the rows, columns and diagonals, "Rows go across, columns go down and diagonals follow oblique lines." (Be sure they understand.) "Look at the third row." "What can you say about the numbers?" "What can you say about column seven?" "Do you see a relationship of the numbers?"

·	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Beginning with one in the upper left corner, circle the diagonal numbers. Ask, "How are the circled numbers related to the numbers in the first row? in first column?" "What is the product when one factor is zero?" "Look at the top row of numbers. Multiply the number in one row by the number in each column. Cross out the numbers of 7. Underline the multiples of 2." Show how the commutative property of multiplication applies to the number pairs. (3 in column 1, and 5 in row 1, also 8 in column 1, and 7 in row 1.)

Look at 3 in column 1, and 2 in row 1. "What is the product?" Look at 5 in row 1, and 4 in column 1. "What is the product?" "Are the products odd or even numbers?" Look at 4 in row 1, and 6 in column 1. Look at 8 in row 1, and 2 in column 1. "Is the product an odd number or an even number?" "What can you say about (a) the product of two even numbers?, (b) the product of two odd numbers?, (c) the product of an odd and an even number? "Circle with a red marking pencil all the even numbers in the table." "Circle with a blue marking pencil all of the odd numbers in the table." Ask, "What is the interval between numbers in the vertical column?" Ask, "What is the interval between numbers in the diagonal column?"

obj. 9

2. As a device for further illustrations of operations with whole numbers, the teacher may make mimeograph copies of the calendar such as the following.

A 7's Calendar

Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

Ask pupils to study the arrangement of the numbers looking for patterns. Ask in particular for the pattern for the numbers in the first column; the pattern for column four; the pattern for a diagonal. Twenty-eight minus what number equals 22? Twenty-two minus what number equals 16? 16 − ? = 10. 10 − ? = 4. Ask pupils to discover other patterns. Also make other calendars.

obj. 14

3. The teacher may illustrate the magic square using the illustration below.

Magic square using numbers 1-9

4	3	8
9	5	1
2	7	6

Find the sum of each row of numbers; of each column of numbers; of each diagonal line of numbers. Ask if the sum was the same in every case. Make duplicate copies of squares. Ask pupils to try rearranging the numbers and add again. What happens?

Use a magic square using numbers two through ten, such as the following.

9	4	5
2	6	10
7	8	3

What is the sum of each row? Each column? Each diagonal? Add in opposite directions. What happens?

Illustrate by using the overhead projector or draw on the chalkboard a 4 by 4 magic square using numbers 1 through 16 as follows.

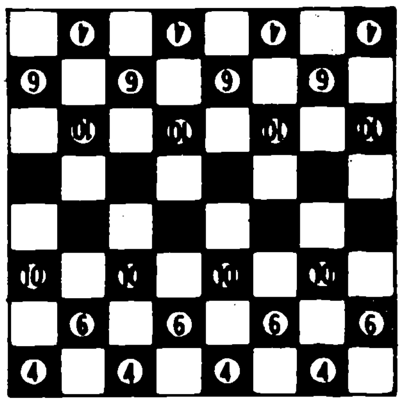
16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Ask pupils to find the sum of the numbers in row two, the sum of the numbers in column three and the sum of the numbers on the diagonal beginning with 16. Reverse the order of the addition. What happens?

Ask pupils to draw other 4 by 4 squares using numbers 1 through 16. Is the figure thus formed a magic square?

obj. 14

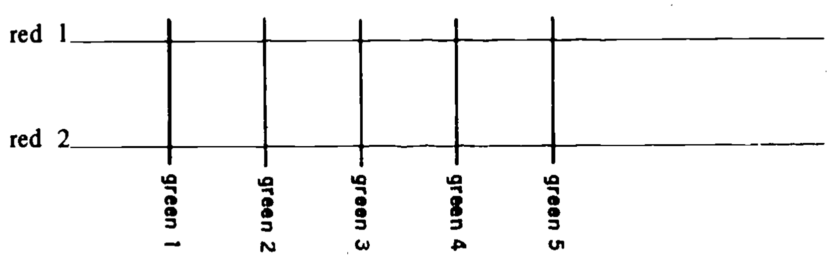
4. A game of checkers may be used to reinforce number combinations. Regular rules for checkers may be used; however, the red and black discs are numbered as shown on the diagram. Pupils score this game by adding the numerals on the checkers as jumps are made. For example, if a red 4 jumps a black 6, the red scores 10. A double jump would cause the pupil to add three numbers. When a checker reaches the King row, it doubles in value. The player with the highest total score wins. Different numerals may be used according to the drill that is needed by pupils. This game may be adapted to multiplication, also.



obj. 1

5. Another model for a Cartesian product may be formed by using strings of two different colors.

Example



Pairing 2 red strings with 7 green strings gives 14 combinations. Ask the pupils to name the points of intersection by stating ordered pairs of the strings. Some examples are

(Red 1, Green 1), (Red 1, Green 2), (Red 1, Green 3), etc.

obj. 2

6. Prepare a worksheet or transparency using the following suggestions.

a. Circle each of the even numbers in set A.
A = { 6, 9, 10, 4, 11, 12, 15, 16, 17, 20 }

b. Express each of the even numbers in the set as the product of 2 and some counting number. For example, 6 = 2 X 3

Pupils should work together to solve problems using concrete materials. They should be encouraged to discuss the question "Do you think every even whole number can be expressed as 2 times some whole number?"

obj. 2,3

7. Develop a class discussion around the following suggestions.

   a. Circle all of the odd numbers in set B.
      $B = \{1, 2, 3, 5, 6, 7, 9, 10, 11, 12, 14, 15\}$

   b. Express each odd number in set B as the sum of an even number ar . For example, 3 = 2 + 1

   c. Can you find an odd number which is not one more than some even number?

obj. 3

8. Prepare ditto sheets with problems such as those suggested below.

   a. Write numerals in the frames to make true sentences.

$2 + 2 = \square$	$2 + 6 = \square$
$0 + 8 = \square$	$4 + 4 = \square$
$4 + 2 = \square$	$6 + 4 = \square$

   b. Are all the addends even or odd?

   c. Are all the sums even or odd?

   d. Can you find two even numbers whose sum is an odd number?

obj. 2,3

9. Prepare ditto sheets or a transparency to use as a basis for class discussion.

   a. Write numerals in the frames to make true sentences.

$1 + 3 = \square$	$5 + 3 = \square$
$5 + 1 = \square$	$5 + 5 = \square$
$3 + 3 = \square$	$1 + 5 = \square$

   b. All the addends are (odd, even) numbers.

   c. All the sums are (odd, even) numbers.

   d. Can you find two odd numbers whose sum is an odd number? Try!

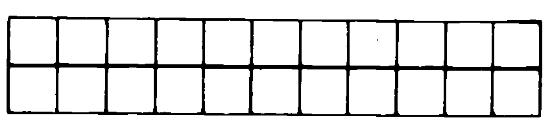
   e. Can you guess which seems to be true from the above observations?

   f. Within a single sentence all frames of the same shape represent the same number. Classify each expression as even or odd, if possible, and justify your answer in each case:

$\square + \square$	$\triangle + \triangle + \triangle + \triangle$
$\triangle + \triangle + 1$	$\triangle + \triangle + \triangle$

obj. 2,3

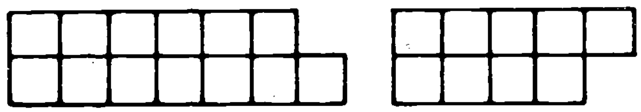
10. a. Cut strips of light weight cardboard. (Old manila folders are a good weight.) Make strips 2 by *n*, where *n* is any natural number.

2 

11

This strip represents the even number 22, 2×11. By folding accordian fashion it can be used to represent any even number, $2 \times n$. Another such strip may be used to represent any other even number, $2 \times m$, where m is any natural number. By joining the two strips end to end, pupils can demonstrate that the sum of any two even numbers is an even number.

b. Make representations of two odd numbers from strips of paper as described above.



$(2 \times 6) + 1 = 13$ $\qquad$ $(2 \times 4) + 1 = 9$

Let pupils use these models to show that 13 + 9 represents an even number. By folding accordian fashion these models can be used to represent any two odd numbers less than 15 and 11, respectively, and can serve to help pupils realize that the sum of *any* two odd numbers is an even number.

obj. 2,3 11. Present questions such as the following to pupils to encourage them to generalize the results of operations with even numbers and odd numbers.

a. Is the sum of any two even numbers even or odd?
b. Is the sum of any two odd numbers even or odd?
c. Is the sum of an odd number and an even number odd or even?
d. Is the product of two even numbers even or odd?
e. Is the product of two odd numbers even or odd?
f. Is the product of an odd number and an even number even or odd?

After some discussion encourage students to make charts of sums and products of whole numbers and to study the results in order to help them make generalizations.

Examples

For question (a).

+	2	4	6	8	. . .
2					
4					
6					
8					
⋮					

For question (f).

×	2	4	6	8	10	. . .
1						
3						
5						
7						
9						
⋮						

obj. 4

12. Use a transparency or ditto copies of a hundreds square, a 10×10 grid. Begin with the number 1 and numbering across to the right and back again from left to right label the squares 1-100.

The Sieve of Eratosthenes is explained in most textbooks as a technique for finding prime numbers less than a given whole number, usually 100. Hence a detailed account is not needed here. The following steps summarize the method.

a. Cross out 1 because it is not prime by definition.
b. Cross out all multiples of 2 except 2.
c. The next three prime numbers in order are 3, 5, 7. Cross out all multiples of 3, 5, 7, except 3, 5, 7.
d. Circle the remaining numbers. They are prime numbers less than 100.

It is not necessary to instruct pupils to cross out multiples of 4 or 6 because they were crossed out as multiples of 2 and of 3.

Discuss why you need not continue the steps above to include crossing out all multiples of 11, the next prime number. (Any number less than 100 which has a factor of 11 has another factor less than 11 and hence has already been crossed out.)

Make a chart of all prime numbers less than 100, and display it in the room.

obj. 3,4

13. Develop a class discussion around the following questions.

a. Circle the prime numbers in each set.

$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$B = \{2, 4, 6, 8, 10, 12\}$
$C = \{1, 3, 5, 7, 9, 11, 13, 15\}$

b. Can you find a prime number that is an even number?
Can you find an odd number which is not prime?
Can you find two consecutive numbers that are prime?
Can you find three consecutive odd numbers that are prime?

c. Tell whether each of the following statements is true or false.

1. All even numbers are composite.
2. All odd numbers are prime numbers.
3. One is a prime number.

d. How many pairs of twin primes can you find? (Twin primes are primes whose difference is 2.) For example, $5 - 3 = 2$. Hence 3 and 5 are twin primes.

obj. 5

14. Ask at least 5 pupils to factor the same number. Choose numbers appropriate for the pupils' abilities from the set given.

$\{54, 75, 120, 168, 225, 363, 432, 576\}$

Compare the five factorizations of each number. Were the factorizations all the same? If not, how were they different? Discuss the fact that the order of factors makes no difference in the product, hence the pupils should suspect that every composite whole number can be expressed as a product of primes in just one way (except for order.) This fact is called "The Fundamental Theorem of Arithmetic."

obj. 4

15. Can you write names for all even numbers 14 or greater as sums of two prime numbers in more than one way?

Example

$14 = 7 + 7 = 3 + 11$

$16 = 3 + 13 = 11 + 5$

$18 = 5 + 13 = 7 + 11$

$20 = \_\_ + \_\_ = \_\_ + \_\_$

etc.

obj. 5,7

16. Ask pupils to construct a factor tree for each of the numbers of a set such as Set A.

$A = \{ 24, 36, 42, 417, 225, 266 \}$

Of course, immediate recall of basic facts is helpful in factoring 24, 36 and 42, but for 225 and 417 new techniques will be helpful. How can a pupil find a factor to begin with? The committee recommends the divisibility tests for 2, 3, 5, 9 (and 7, if so desired).

The pupil should already know that all even numbers are divisible by 2 so he needs only to look at the ones digit to tell if the number is divisible by 2.

Divisibility tests should be discovered by the pupils, when possible, rather than be given to them as rules.

Divisibility by 3 can be determined by seeing if the sum of the digits is divisible by 3. The number 261 is divisible by 3 since $2 + 6 + 1 = 9$ and 9 is divisible by 3.

A whole number is divisible by 4 if and only if the last two digits (the ones and the tens) represent a number divisible by 4.

Pupils probably already know that a number is divisible by 5 if and only if the ones digit is 5 or 0.

Divisibility of a number by 9 may be determined by seeing if the sum of the digits is divisible by 9. 621 is divisible by 9 since $6 + 2 + 1$ is divisible by 9.

The following test for divisibility by 7 is interesting and it is easy enough to be practical. Many tests for divisibility by 7 are not.

Is 266 divisible by 7?

$$\begin{array}{rl} 26 & 6 \\ -12 & \\ \hline 14 & \end{array}$$

Isolate the ones digit.

Double the ones digit and subtract it from the remainder of the number beginning with the tens digit. If the remainder is not a one or two digit number repeat the process. The original number is divisible by 7 if and only if the resulting one or two digit number is divisible by 7.

From this we would conclude that 266 is divisible by 7 since 14 is divisible by 7.

Is 1667 divisible by 7?

$$\begin{array}{rl} 166 & 7 \\ -14 & \\ \hline 15 & 2 \\ 4 & \\ \hline 11 & \end{array}$$

Since 11 is not divisible by 7, 1667 is not divisible by 7. Proofs of some divisibility tests can be found in references listed in the bibliography. Capable seventh and eighth grade students should be encouraged to see why these tests work.

obj. 7

17. a. Have pupils fill in the blank for the *ones* in each numeral to make a number that is divisible by the number given.

Example

(1) Some numbers divisible by 2: { 14, 28, 36, 40, 54 }
Is there a unique answer in each case?

(2) Some numbers divisible by 3: { 2_, 3_, 5_, 7_, 8_ }
Is there a unique answer in each case?

(3) Some numbers divisible by 4: { 32_, 21_, 13_, 24_ }
Is there a unique answer in each case?

(4) Some numbers divisible by 9: { 17_, 30_7, 154_8, 300_ }
Is there a unique answer in each case?

b. Beside each number, write 2 if the number is divisible by 2; if the number is divisible by 3, write 3; if divisible by 4, 5, 7, 9 write the appropriate number. If the number is not divisible by 2, 3, 4, 5, 7 or 9 write "no".

(1)	105	(6)	219
(2)	42	(7)	51
(3)	300	(8)	75
(4)	187	(9)	87
(5)	96		

obj. 9

18. An interesting way to write all possible product expressions involving exactly two factors for a counting number is to list all of the factors of the number in increasing order. Then pair the first factor with the last one, the second factor with the second from the last, and so on. For example, find all possible two-factor product expressions for the number 105. The factors are

1, 3, 5, 7, 15, 21, 35, 105.

The desired product expressions are $1 \times 105, 3 \times 35, 5 \times 21, 7 \times 15$.

Ask pupils to find all possible two-factor product expressions of the following.

a. 8 c. 100
b. 18 d. 275

obj. 9

19. Help pupils generalize that the nth square number is $n \times n$ or n^2.

Example

1st square number is 1.
2nd square number is 4. $4 = 2^2$
3rd square number is 9. $9 = 3^2$
4th square number is _______.
.
.
.
10th square number is _______.
nth square number is ________.

Have the pupils list the set of square numbers to 100 and then discuss the following questions.

a. Is there a pattern of odd and even square numbers?

b. See if there is a square number greater than 1 which is a divisor of each even square number.

c. What is the result if any odd square number is divided by 4?

d. Is there a pair of square numbers whose sum is also a square number?

e. Is there a square number that is twice as large as another square number; three times as large, four times as large, nine times as large? Can you explain this?

f. Use the list of square numbers to 100 and study the differences between successive terms.

$$\left\{ \begin{matrix} 0, & 1, & 4, & 9, & 16, & 25, & 36, & 49, & 64, & 81, & 100 \\ & 1 & 3 & 5 & ? & \ldots & & & & & \end{matrix} \right\}$$

Can you state what you observe?

obj. 5 20. Have pupils find the sum of the divisors of 6, other than 6 itself. (1 + 2 + 3 = 6.) Find another counting number that is equal to the sum of the divisors that is less than the number itself. The Greeks called such numbers *perfect numbers.*

A perfect number is one that equals the sum of its proper divisors, and the proper divisors of a number are all those except the number itself.

The first four perfect numbers are 6, 28, 496, 8128. Note that the first perfect number is a single digit numeral, the second is a two digit, the third is a three digit, and the fourth is a four digit – all in base ten numeration. However, the fifth perfect number has eight digits, 33550336.

Ask groups of pupils to confirm the fact that these five numbers are perfect numbers.

Arrange the numerals vertically and study the digits. What pattern do you observe in the last digits of the numerals? Only twenty-three perfect numbers have been found. The largest of these numbers requires 6800 decimal digits to write the numeral.

Other interesting ideas concerning the perfect number may be found in books listed in the annotated references in the section Utilization of Media.

obj. 9 21. Continue squaring the numbers given below until you can discover the pattern, then predict the answer to the next problem. Check to see if your guess is correct. Do you see why this works?

$$\begin{aligned} 1^2 &= 1 \\ 11^2 &= 121 \\ 111^2 &= \\ 1111^2 &= \\ 11111^2 &= \end{aligned}$$

Complete these multiplications to find some interesting patterns.

7 X 7 =	4 X 4 =
67 X 67 =	34 X 34 =
667 X 667 =	334 X 334 =
6667 X 6667 =	3334 X 3334 =
	33334 X 33334 =

obj. 8

22. Ask pupils to do the following.

a. Write the set of natural number multiples of 4, of 6, of 8, that are less than 100.

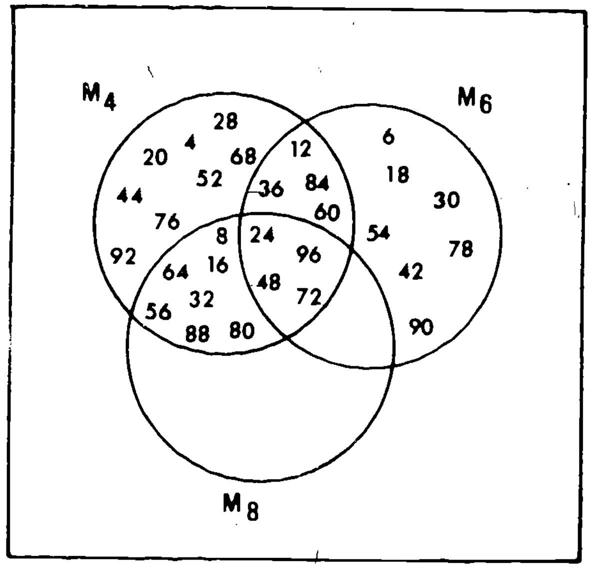
$M_4 = \{ 4, 8, 12, \_\_, \_\_, \text{etc.} \}$

$M_6 = \{ 6, 12, \_\_, \_\_, \_\_, \text{etc.} \}$

$M_8 = \{ 8, 16, 24, \_\_, \_\_, \_\_, \text{etc.} \}$

b. Show the common multiples of M_4, M_6 and M_8 using Venn Diagrams.

Solution



c. Find the least common multiple of 4, 6 and 8, that is LCM (4,6,8) = ________

d. List the members of each

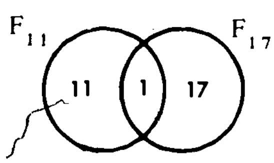
a. $M_4 \cap M_6$

b. $M_6 \cap M_8$

c. $M_4 \cap M_6 \cap M_8$

obj. 7

23. a. Ask the pupils to find the greatest common factor of 11 and 17.



Since $F_{11} \cap F_{17} = 1$, 1 is the greatest common factor, in fact, the only common factor.

If GCF (a,b) = 1, a and b are whole numbers, then a and b are relatively prime.

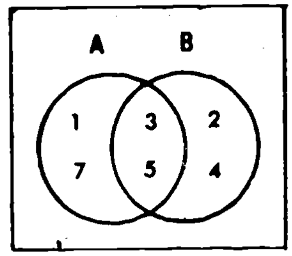
Circle the pairs of numbers that are relatively prime and tell why they are or are not relatively prime.

(3,5), (2,8), (3,27), (49,12), (3,8), (7,17).

b. Ask the pupils to draw Venn diagrams to show the intersection of the sets.

$A = \{1, 3, 5, 7\}$
$B = \{2, 3, 4, 5\}$

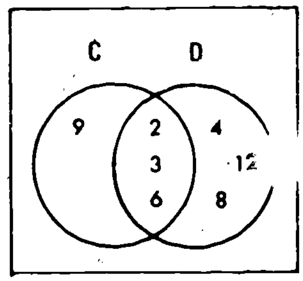
Solution



c. Ask the pupils to draw the Venn diagrams to show the intersection of the sets.

C is the set of factors of 18
D is the set of factors of 24

Solution

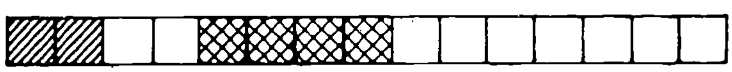


Name the greatest common factors of 18 and 24.

Rational Numbers: Pre-addition

obj. 11,12b

24. Each pupil should have a sheet of graph paper. Outline fifteen squares in a row to be the unit, as shown below. Have the pupils color $\frac{2}{15}$ of the strip red and $\frac{4}{15}$ of the strip green and leave two separate sections of the strip uncolored. Ask what pair of numbers names the colored part of the unit ($\frac{6}{15}$, since 6 of the 15 equal parts are colored.) Repeat, using other combinations such as $\frac{3}{15}$ with $\frac{1}{15}$, $\frac{7}{15}$ with $\frac{5}{15}$. In each case the fraction name (not the rational equivalent) should be given.



obj. 11,12b

25. Use a strip of 12 contiguous squares as the unit and repeat the previous activity using combinations such as $\frac{5}{12}$ with $\frac{2}{12}$, $\frac{2}{12}$ with $\frac{5}{12}$, $\frac{3}{12}$ with $\frac{2}{12}$, and $\frac{2}{12}$ with $\frac{3}{12}$. The fraction name illustrated should always be given.

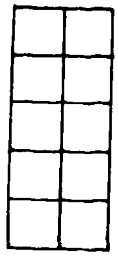
obj. 11,12b

26. Have the pupils take as a unit a region consisting of 2 rows of 4 squares each, as shown below.

Since the natural partition of this unit is eighths, the following combinations would be appropriate: color $\frac{3}{8}$ red and $\frac{1}{8}$ green; $\frac{1}{8}$ red and $\frac{3}{8}$ green, etc. In each case, give the fraction name for the part of the unit which is colored.

obj. 11,12b

27. Use as a unit a region consisting of 5 rows of 2 squares each and such combinations as $\frac{4}{10}$ with $\frac{2}{10}$, and $\frac{2}{10}$ with $\frac{4}{10}$, etc.

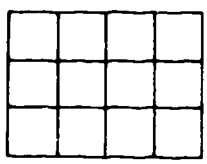


Rational Numbers: Addition

obj. 11,12b, 15

28. Addition of rational numbers can be derived as an extension of the previous activity, again using graph paper. The following sequence suggests the way in which some addition examples may be developed.

a. Take as a unit a region consisting of 3 rows of 4 squares each.



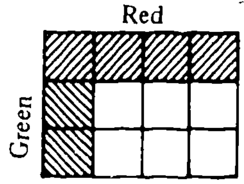
Since there are 3 rows all the same size, each row represents $\frac{1}{3}$ of the unit. Have each pupil color $\frac{1}{3}$ of the unit, noting that any row will do.

b. Draw another unit just like the one for part a. Since there are four columns all the same size, each column represents $\frac{1}{4}$ of the unit. Have each pupil color $\frac{1}{4}$ of the unit, noting that *any* of the four columns will do.

c. Next have the pupils outline several units exactly like those in parts a and b; this time they are to color $\frac{2}{12}$ of the unit in each picture, and these are all to be different.

d. When the pupils can accept that there are many ways to color $\frac{2}{12}$ of the unit, and at least three ways to color $\frac{1}{3}$ of the unit, they are ready for the next step.

Have the pupils use a unit just like the others and color $\frac{1}{3}$ of the unit red, then $\frac{2}{12}$ of it green, making sure that no part is colored twice. It is not necessary, of course, for all the pupils to have the same picture.

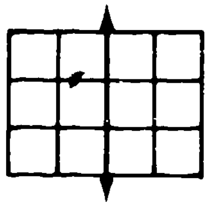


Next, ask the pupils to name the part of the unit which is colored. This part is obviously $\frac{6}{12}$, and this fraction name, rather than the rational equivalent, should be given.

e. A unit just like the others should again be used, and the pupils asked first to color $\frac{1}{4}$ of the unit red, then $\frac{2}{12}$ of it green, with no part colored twice. The name $\frac{5}{12}$ can obviously be given for that part of the unit which is colored.

f. The results of parts d and e are to be recorded using symbols in the problem, that is, for d $\frac{1}{3} + \frac{2}{12} = \frac{6}{12}$ and e $\frac{1}{4} + \frac{2}{12} = \frac{5}{12}$.

g. To extend the above activities again using a three by four unit, the pupils should be asked to color $\frac{1}{2}$ of it. There is a *natural* partition down the middle which can be used and either half could be colored.



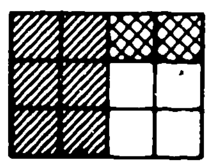
If by now some pupils realize that any six of the twelve parts could be construed as one-half, they should be free to use this knowledge, but others should not be asked to accept this idea at this time. When everyone has colored $\frac{1}{2}$ of the unit, they should color an additional $\frac{2}{12}$ and name the colored part. In symbols $\frac{1}{2} + \frac{2}{12} = \frac{8}{12}$. Repeat, using such combinations as $\frac{1}{2} + \frac{1}{12}$, $\frac{1}{12} + \frac{1}{2}$, $\frac{1}{2} + \frac{5}{12}$, $\frac{5}{12} + \frac{1}{2}$.

obj. 11,12b, 15

29. Outline another three by four unit and ask the pupils to find a natural partition into sixths. By this time, they should be familiar enough with the characteristics of this particular unit that they can find such a natural subdivision.

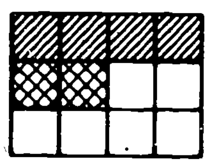
From the parts a, b, g of the previous activity and the present one, the following sums can be computed. It is probably wise to give the fraction name to the result in all cases.

$\frac{1}{2} + \frac{1}{6} = \square$



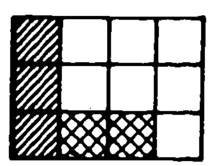
$\left\{\frac{8}{12}\right\}$ or $\left\{\frac{2}{3}\right\}$

$\frac{1}{3} + \frac{1}{6} = \square$



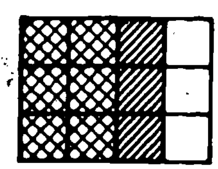
$\left\{\frac{6}{12}\right\}$ or $\left\{\frac{3}{6}\right\}$ or $\left\{\frac{1}{2}\right\}$

$\frac{1}{4} + \frac{1}{6} = \square$



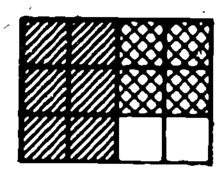
$\left\{\frac{5}{12}\right\}$

$\frac{1}{2} + \frac{1}{4} = \square$



$\left\{\frac{9}{12}\right\}$ or $\left\{\frac{3}{4}\right\}$

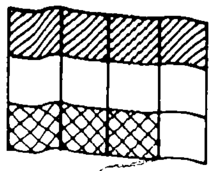
$\frac{1}{2} + \frac{2}{6} = \square$



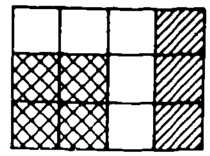
$\left\{\frac{10}{12}\right\}$ or $\left\{\frac{5}{6}\right\}$

obj. 11,12b, 15

30. Brain teaser – Can anyone show $\frac{1}{3} + \frac{1}{4}$? The problem here, of course, is that the *natural* partitions for $\frac{1}{3}$ and $\frac{1}{4}$ would overlap and the solution *must* come from the realization that any 3 of the 12 parts can be called $\frac{1}{4}$ or that any 4 of the 12 parts can be called $\frac{1}{3}$. Thus the pupil who proposes a solution will have to demonstrate some variation of one of the following.



or



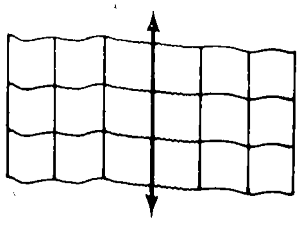
obj. 11,12b, 15

31. It may now be possible to relate these activities to pencil-and-paper activities by recalling equivalence classes, since

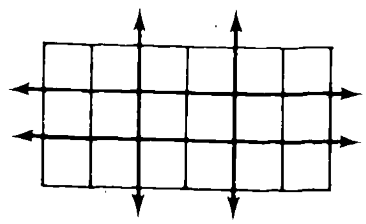
$$\frac{1}{3} \in \left\{ \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots \right\}$$

and $$\frac{1}{4} \in \left\{ \frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \ldots \right\}$$

by choosing $\frac{4}{12}$ as the name $\frac{1}{3}$, and $\frac{3}{12}$ as the name for $\frac{1}{4}$, $\frac{1}{4} + \frac{1}{3}$ may be replaced by $\frac{3}{12} + \frac{4}{12}$ and named $\frac{7}{12}$. However, some pupils may need additional activities of the type outlined in a through e of Activity 28. Some other possible units made from graph paper are the three by six, exhibiting natural partitions of 2, 3, 6, 9 and 18.



2 parts



9 parts.

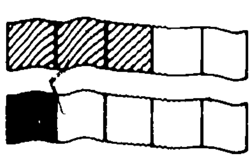
Also, other units to use are the two by five (halves, fifths and tenths being obvious); the three by five (thirds, fifths and fifteenths being obvious); and, as comparison, a three by eight and four by six.

Rational Numbers: Pre-subtraction

There are several types of situations which must be classified or associated with a mathematical operation called subtraction of rational numbers. The following sequence of activities shows how these can be presented with fractions.

obj. 11,12b, 15

32. Each pupil should have a piece of graph paper and take as a unit five squares in a row. Two such units should be outlined, one slightly below the other so that they can be seen to be the same size. Color $\frac{3}{5}$ of the top unit and $\frac{1}{5}$ of the bottom one.

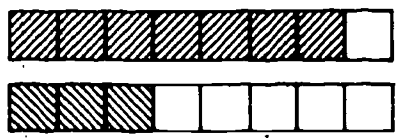


Obviously, $\frac{3}{5}$ of the unit is larger than $\frac{1}{5}$ of the unit. Two questions can be asked. "What must be added to $\frac{1}{5}$ to get $\frac{3}{5}$?" (Answer– $\frac{2}{5}$) or, "$\frac{3}{5}$ of this unit is how much larger than $\frac{1}{5}$ of this unit?" (Answer– $\frac{2}{5}$). The answers to these questions lead us to say that the difference of $\frac{3}{5}$ and $\frac{1}{5}$ is $\frac{2}{5}$. The pupils might imagine $\frac{1}{5}$ of the unit to be taken away from the $\frac{3}{5}$ shown in the upper diagram and asked to name the part that was left. Again this is $\frac{2}{5}$. If the results of this exploration are symbolized, we get either $\frac{1}{5} + \square = \frac{3}{5}$, with $\frac{2}{5}$ the replacement for $\square$, or $\frac{3}{5} - \frac{1}{5} = \square$, with $\frac{2}{5}$ the replacement for $\square$.

These questions reflect the various situations associated with subtraction—in order, the inverse of addition, comparative subtraction and take-away subtraction. At this level, the students' previous experience with all of these types of subtraction within the framework of the system of whole numbers is relied on.

obj. 11,12b, 15

33. It may be advisable to repeat the previous activity with a different unit, for example, a strip of eight squares. Outlining two such units, color $\frac{7}{8}$ of the top one and $\frac{3}{8}$ of the bottom one.



Again ask, "What must be added to $\frac{3}{8}$ to get $\frac{7}{8}$?" "$\frac{7}{8}$ is how much larger than $\frac{3}{8}$?" "If $\frac{3}{8}$ is taken from $\frac{7}{8}$, name the part that is left." Since answers to all of these questions are the same ($\frac{4}{8}$), we write $\frac{7}{8} - \frac{3}{8} = \frac{4}{8}$.

Use as many other examples (different units, different parts) as necessary so that the students can name differences of pairs of fractions without recourse to the model.

Rational Numbers: Subtraction

obj. 11,12b, 15

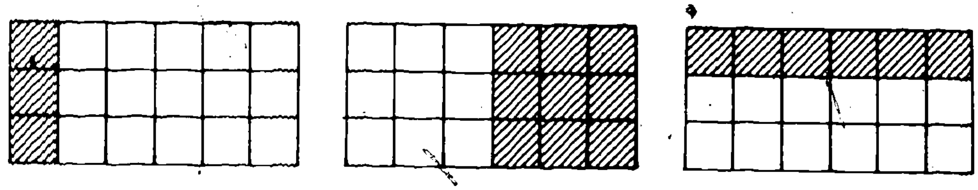
34. To introduce subtraction of rational numbers a unit consisting of three rows of six squares is useful. Again, graph paper is convenient for the students to use. Have each student draw two such units and color $\frac{1}{3}$ of each unit. Two different partitions into thirds should be shown by each pupil. Have each pupil color $\frac{1}{18}$ of the unit on one of their diagrams, using a different colored crayon. Then ask, "How should we name $\frac{1}{3} - \frac{1}{18}$?" It should be noted that it makes no difference here whether the $\frac{1}{18}$ is part of the portion colored as $\frac{1}{3}$ or the portion not colored. If the $\frac{1}{18}$ is part of the original $\frac{1}{3}$, the subtraction could be interpreted as either take away or comparison; whereas, if the $\frac{1}{18}$ is selected outside the part colored first, the subtraction can be interpreted as either comparison or as the inverse of addition. In the later case, the $\frac{1}{18}$ will belong to some other third of the unit. The answer in any case is $\frac{5}{18}$. Because of the fact that there are several possible ways to arrive at the $\frac{5}{18}$, individual pupils should be asked to explain how they got their answers. Next, using the other unit, have each student color $\frac{4}{18}$ of it. Then ask, "How shall we name $\frac{1}{3} - \frac{4}{18}$?" Having agreed on $\frac{2}{18}$, students should be encouraged to explain various methods of arriving at this answer.

obj. 11,12b, 15

35. Have each pupil draw another unit consisting of three rows of six, and this time, finding a natural partition, color $\frac{1}{2}$ of this unit. Next, each pupil should color $\frac{1}{18}$ of the unit and ask, "How shall we name $\frac{1}{2} - \frac{1}{18}$?" When everyone is satisfied that $\frac{8}{18}$ is appropriate, ask for a name for $\frac{1}{2} - \frac{4}{18}$ without using a model if possible.

obj. 11,12b, 15

36. Have each pupil outline three units, each a 3 by 6 region, and ask, "How many rows are in one of these units?" then "How many columns are in one of the units?" Then ask the pupils if they see a natural partition into sixths. A column seems the obvious choice, and each pupil should color $\frac{1}{6}$ of the first unit. The second unit should show $\frac{1}{2}$, and the third unit $\frac{1}{3}$. A sample might look like this.



$\frac{1}{6}$ $\qquad$ $\frac{1}{2}$ $\qquad$ $\frac{1}{3}$

Questions

How could we name $\frac{1}{2} - \frac{1}{6}$? ($\frac{6}{18}$ is an acceptable answer)

How could we name $\frac{1}{3} - \frac{1}{6}$? ($\frac{3}{18}$ is an acceptable answer but $\frac{1}{6}$ might be expected.)

How could we name $\frac{1}{2} - \frac{1}{3}$? (either $\frac{3}{18}$ or $\frac{1}{6}$ is acceptable)

If the pupils seem to need more practice, a two by five unit could be used. In different models, color $\frac{1}{2}$, for which there is a natural partition. For example, use a model with ten units and then ask, "How shall we name $\frac{1}{2} - \frac{1}{10}$?" Repeat for $\frac{1}{2} - \frac{3}{10}$. Another natural partition shows fifths and some related problems would be naming $\frac{1}{5} - \frac{1}{10}$, $\frac{2}{5} - \frac{1}{10}$, $\frac{2}{5} - \frac{3}{10}$. Further practice with other units and other problems may be desirable before proceeding to the activity numbered 36.

obj. 11,12b, 15

37. After the class has completed the above activities, guide the pupils to relate their work here to previous work with equivalence classes.

$$\frac{1}{2} \in \left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots, \frac{9}{18} \ldots \right\}$$

$$\frac{1}{3} \in \left\{ \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \ldots, \frac{6}{18} \ldots \right\}$$

$$\frac{1}{6} \in \left\{ \frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \ldots \right\}$$

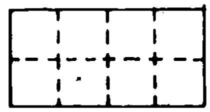
Thus $\frac{1}{2} - \frac{1}{6}$ can be named $\frac{9}{18} - \frac{3}{18} = \frac{6}{18}$,
and $\frac{1}{3} - \frac{1}{6}$ can be named $\frac{6}{18} - \frac{3}{18} = \frac{3}{18}$,
and $\frac{1}{2} - \frac{1}{3}$ can be named $\frac{9}{18} - \frac{6}{18} = \frac{3}{18}$.

Rational Numbers: Multiplication

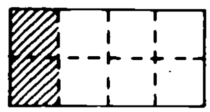
The physical model or situation for multiplication is very often not clear. The pupil has a fixed notion of multiplicative situations with whole numbers. This notion does not make sense for rational numbers. One way to make the product of rational numbers reasonable is to let the pupil find *for himself* an acceptable answer to certain problems before identifying these as multiplication problems. In the activities below, unit regions and their fractional parts are used to develop the interpretation of multiplication of rational numbers.

obj. 11,12b

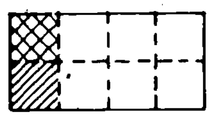
38. As an example of an activity, a unit should be selected. Graph paper is a must for this purpose. Have each pupil outline a region consisting of two rows of four squares each. This will be the unit.



Ask the pupil to color $\frac{1}{4}$ of this unit. The following drawing is shaded to illustrate one possible choice.

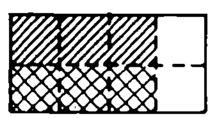


Next have each pupil color $\frac{1}{2}$ of the colored part. The following illustrates a completion.



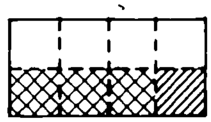
Now ask, "What part of the unit have you colored twice?" The pupils should see that this is one part out of eight or $\frac{1}{8}$ of unit. Then, "What is $\frac{1}{2}$ of $\frac{1}{4}$ of a whole?"

39. a. Have each pupil outline another two by four unit and color $\frac{3}{4}$ of this unit. Then have them color $\frac{1}{2}$ of the part they just colored. The result might look like this.



Ask "What part of the unit is colored twice?" Then, "What is $\frac{1}{2}$ of $\frac{3}{4}$?"

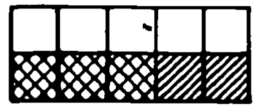
b. Using another two by four unit, have each pupil color $\frac{1}{2}$ of the unit and then $\frac{3}{4}$ of what they just colored. The results might look like this.



Ask "What part of the unit has been colored twice?" Then ask "What is $\frac{3}{4}$ of $\frac{1}{2}$ of a whole?"

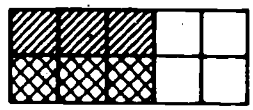
c. Have the pupils compare their pictures for activities b and c. Ask if anyone got exactly the same part colored for both activities. In drawings like those above, the left hand three squares in the bottom row were colored. Some pupils will have the same block colored for both, others will not. Ask if one could have exactly the same blocks colored if one finds $\frac{1}{2}$ or $\frac{3}{4}$ and one finds $\frac{3}{4}$ of $\frac{1}{2}$. Discuss. Point out that the same answer was obtained in each case. Ask if this seems reasonable. Ask a pupil who understands to explain if some of the pupils do not understand.

40. a. Use as a unit a region consisting of two rows of five. Again ask, "How may rows?" and "How many columns?" Ask what part of the unit one row would represent and what part of the unit one column would represent. Have each pupil color $\frac{1}{2}$ of the unit. Then ask each to color $\frac{3}{5}$ of the part he just colored. The result might be as follows.



Ask what part of the unit is colored twice, that is $\frac{3}{10}$.

b. Have each pupil use a two by five unit and color $\frac{3}{5}$ of the unit. Then have each color $\frac{1}{2}$ of the part he just colored. Ask what part of the unit is colored twice. It will be $\frac{3}{10}$ as illustrated below.



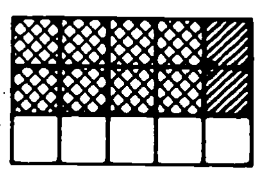
c. Ask if anyone has the same three squares colored in a as he did for b. Ask if this could happen or if it seems reasonable. Ask the students to note that the answers were the same. It should also be pointed out that any three of the squares would represent $\frac{3}{10}$ of the unit; however, the main idea here is that the same three squares can represent $\frac{3}{5}$ or $\frac{1}{2}$ or $\frac{1}{2}$ or $\frac{3}{5}$.

obj. 11,12b

41. Have each child outline another two by five region for a unit, and color 1/2 of it. Next, color $\frac{4}{5}$ of the part just colored.

Ask what the squares colored twice could represent. Have a student show how these four squares could represent $\frac{1}{2}$ of $\frac{4}{5}$ or $\frac{4}{5}$ of $\frac{1}{2}$. The pupils should see that they are the same and that both represent $\frac{4}{10}$ of the unit.

For the next unit, use a region consisting of three rows of five squares. Ask what part of the unit one row might represent; what part of the unit one column might represent. Have each pupil show $\frac{4}{5}$ of $\frac{2}{3}$ on this unit. If the class has trouble, revert to the directions of earlier parts of this activity (color $\frac{2}{3}$ of the unit, then color $\frac{4}{5}$ of the part just colored.) The result might look like this.



Ask what part of the unit has been colored twice ($\frac{8}{15}$). Ask if this same twice-colored part could represent $\frac{2}{3}$ of $\frac{4}{5}$. If necessary ask a pupil to demonstrate that $\frac{4}{5}$ of $\frac{2}{3}$ and $\frac{2}{3}$ of $\frac{4}{5}$ could be represented by the same squares.

If further practice is needed, the three by five unit can be used with $\frac{1}{5}$ of $\frac{1}{3}$, $\frac{1}{5}$ of $\frac{2}{3}$, $\frac{2}{5}$ of $\frac{2}{3}$, etc.

obj. 11,12b

42. Use a two by four unit, and color $\frac{3}{4}$ of it. Ask for a volunteer to color $\frac{5}{5}$ of the part that was just colored. Then ask if all agree. Discuss if necessary. Ask what $\frac{3}{3}$ of the colored part would be. $\frac{13}{13}$ of the colored part, etc. Summarize the discussion of this part by writing the following on the board.

$\frac{5}{5}$ of $\frac{3}{4}$ is $\frac{3}{4}$.

$\frac{3}{3}$ of $\frac{3}{4}$ is $\frac{3}{4}$ etc.

Extend to $\frac{5}{5}$ of $\frac{2}{3}$ of any unit, $\frac{3}{3}$ of $\frac{2}{3}$ of any unit.

obj. 11,12b, 15

43. Using the results of activities 37-40, record the results on the chalkboard, with pupils referring to their pictures to verify the writing.

$\frac{1}{2}$ of $\frac{1}{4}$ is $\frac{1}{8}$

$\frac{1}{2}$ of $\frac{3}{4}$ is $\frac{3}{8}$

$\frac{3}{5}$ of $\frac{1}{2}$ is $\frac{3}{10}$

$\frac{1}{2}$ of $\frac{4}{5}$ is $\frac{4}{10}$

$\frac{2}{3}$ of $\frac{4}{5}$ is $\frac{8}{15}$

Have the children compare the answers with the *of* situation in each case and see if they notice any particular relationship. Ask if they could figure out the answers just by inspecting the numbers. Ask if anyone could tell, without making a picture, what would be $\frac{2}{3}$ of $\frac{7}{8}$. If no one responds, ask the pupils to study the numbers involved in order to find relationships. As relationships are stated, check each example in the first list. The results of activity 41 should also be checked.

$\frac{5}{5}$ of $\frac{3}{4}$; does $\frac{5 \times 3}{5 \times 4}$ name $\frac{3}{4}$?

$\frac{3}{3}$ of $\frac{3}{4}$; does $\frac{3 \times 3}{3 \times 4}$ name $\frac{3}{4}$? etc.

Two or three other examples should then be given, for instance $\frac{2}{3}$ of $\frac{7}{8}$; $\frac{1}{3}$ of $\frac{1}{2}$; $\frac{3}{4}$ of $\frac{2}{5}$. Then say that the multiplication of numerators and the multiplication of denominators is the multiplication of the rational numbers and is written as follows.

$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$; $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$; $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$, etc.

Also $\frac{3}{3} \times \frac{4}{5} = \frac{4}{5}$; $\frac{13}{13} \times \frac{1}{4} = \frac{1}{4}$; etc.

obj. 11,12b, 15

44. Ask if multiplication for rational numbers is commutative, then refer to activities 39, 40 and 41 in particular to see that not only is this reasonable but also that the computational shortcut reflects this property.

Rational Numbers: Division

Division of rational numbers, as multiplication of rational numbers, is difficult to interpret through suitable physical situations. Division of rational numbers is usually taught strictly as a computational process using the well-known rule. It is, however, possible to present this as an operation which is the inverse of multiplication. The agreement in this case is that, if x and y are rational numbers, $x \times y = z$ if, and only if, $x = z \div y$. This conforms to the accepted use of the symbol "$\div$" in the system of whole numbers, so that it should be familiar to the pupils.

This interpretation of division should be developed through a sequence of problems in which the pupils would apply what they have previously learned.

obj. 11,12b, 15

45. a. Have pupils find the missing factors in such examples as the following.

$$\frac{2}{5} \times N = \frac{8}{15}$$

$$\frac{2}{5} \times N = \frac{8}{10}$$

$$\frac{2}{5} \times N = \frac{12}{15}$$

The pupil should be able to find correct replacement for N from his knowledge of multiplication.

b. Using exercises shown in a, and the understanding of the meaning assigned the symbol "$\div$", the pupils should be able to determine each of the following quotients by inspection.

$$\frac{15}{18} \div \frac{5}{9} = N$$

$$\frac{9}{10} \div \frac{3}{2} = N$$

$$\frac{14}{24} \div \frac{2}{3} = N$$

Additional practice will be needed; therefore, further examples similar to these should be used.

obj. 11,12b, 15

46. A true problem for the pupils arises when a sentence such as the last cited above appears in the following form.

$$\frac{7}{12} \div \frac{2}{3} = N$$

In this form the solution is not obvious. At this point the pupils have at hand all necessary information to solve this problem but may need guidance to discover that they can solve the problem. If no one in the class believes there is a replacement for N which will make the sentence true, the teacher should ask why the pupils think there is no solution. The response to be expected is that 2 does not divide 7. So, the next question should be, "Is $\frac{7}{12}$ the only name for that number?" If further help is needed, ask for some other names for $\frac{7}{12}$; then ask if the use of one

of these other names might help solve the problem. Examination of $\frac{7}{12}$, $\frac{14}{24}$, $\frac{21}{36}$, ... should make the solution obvious to some member of the class who can then demonstrate his way of solving the problem.

Further practice will be needed but the problems must be carefully chosen; some examples increasing in complexity are as follows.

$$\frac{6}{7} \div \frac{1}{2} = N$$

$$\frac{6}{7} \div \frac{2}{3} = N$$

$$\frac{1}{4} \div \frac{2}{5} = N$$

$$\frac{1}{4} \div \frac{2}{3} = N$$

In selecting practice exercises for this procedure, a good scheme would be to use the same dividend in several consecutive examples and to change only the divisor; this scheme focuses attention on the renaming process which is the crux of the matter.

The procedure outlined above provides a method for processing division problems based on concepts already developed. An advantage is that it shows the underlying consistency of mathematics, while at the same time it gives the pupils the opportunity to see how to use what they already know when faced with a new problem. Furthermore, the pupils do not have to be reminded which part to invert as they must remember when using memorized rules. A disadvantage is that the process takes a little longer, since the pupils must choose an appropriate replacement for the dividend from an infinite set of possibilities.

Teachers may find some pupils ready for a discussion of division of rational numbers based on the definition, for $\frac{x}{y}$, $\frac{c}{d}$ rational numbers and $c \neq 0$, that $\frac{x}{y} \div \frac{c}{d} = \frac{x}{y} \times \left(\frac{c}{d}\right)^{-1}$

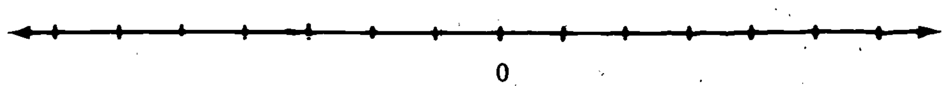
Once division of rationals has been established, it follows, for example, that

$$\frac{7}{8} = \frac{7 \times 1}{1 \times 8} = \frac{7}{1} \times \frac{1}{8} = \frac{7}{1} \div \frac{8}{1}$$

Since $\frac{7}{1}$ and $\frac{8}{1}$ correspond respectively to the whole numbers 7 and 8, say that $\frac{7}{8} = 7 \div 8$. Of course this statement has a logical basis only after division in the rationals has been established, for the symbol "7 ÷ 8" is meaningless in the system of whole numbers.

Integers: Addition and Subtraction

47. Associate the numerals 2, $^-2$, 5, $^-5$, 8, $^-8$, 10, $^-10$ with the appropriate points on the number line below. Name the points in between if you wish.



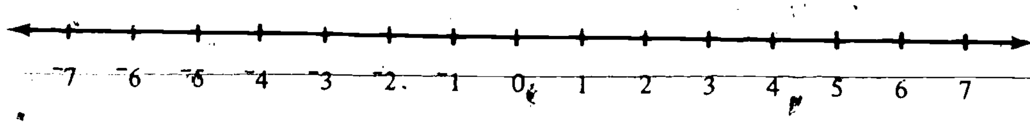
Now do this exercise—Insert the correct symbol between each pair of numbers below. Use >, <, or =.

2	4	$^-4$	4	10	$^-10$
7	3	$^-5$	$^-3$	0	0
2	0	$^-2$	0	0	3

A background for understanding operations involving negative numbers can be established by using the following activities as models.

obj. 12c

48. If one travels a number of miles from one point to another, he may travel part way east and part way west. Using arrows, → means east and ← means west. The arrows may be placed over the numerals.



Example

Mr. Jones travels east four miles and from that point travels west 6 miles. Where is he with respect to his starting point? Two miles west of where he started. The point here is not how far he has traveled, which obviously is 10 miles, but where he is with respect to his starting position.

Many of the same type problems may be used until the pupils discover the rule, such as the following.

(a) If he travels six miles east, then four miles west, what is the distance to the starting point?

(b) If he travels four miles west, then five miles east, what is the distance to the starting point?

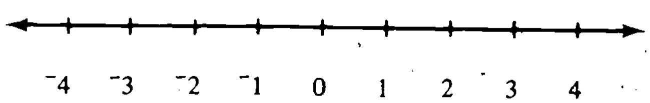
When the teacher thinks the pupils have the idea then she may proceed with the recording. Since we name the associated operation addition in this case, the recording for the first illustration given is $^{+}4 + {}^{-}6 = {}^{-}2$. Read this as "Positive 4 plus negative 6 equals negative 2."

obj. 12c

49. After having learned the addition processes of integers, the pupils may be introduced to the subtraction process as the inverse of addition. The teacher may continue using the east - west idea on the number line as developed in the addition process.

Example

For the number sentence $^{+}2 - {}^{+}3 =$



Using addition, the inverse of subtraction, one would say, "What has to be added to positive three to get to positive 2?" One would obviously have to travel west one place so the missing addend is negative 1.

Enough of these activities may allow the pupils to get a clear understanding of the idea.

Example

$^{-}3 - {}^{-}1 =$

The question is—"What must be added to $^{-}1$ to get to $^{-}3$?" Since the move is two places to the west, the missing addend is $^{-}2$.

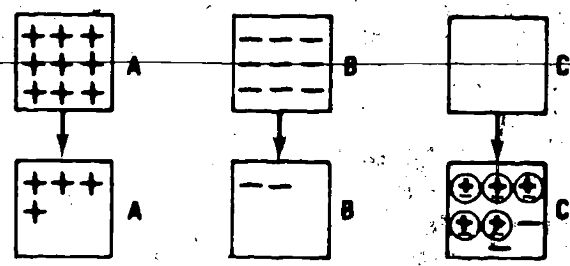
Continue with many similar examples.

obj. 12c

50. Probably at the level when directed numbers are introduced, some pupils will have had experience with positive charges attracting negative charges and neutralizing each other. The teacher may wish to show neutralization with charged particles using a science demonstration.

The teacher could also demonstrate the idea by using an illustration of two buckets, one bucket A, containing an indefinite number of positive particles and the other bucket , B, an indefinite number of negative particles. The question may be asked, what will be the result if seven negative particles from bucket B are placed in an empty bucket, C, and then five positive charges from bucket A are added? Each negative particle will attract a positive particle, and they will neutralize.

This neutralization can be shown by drawing a circle around each pair.



When 5 positive particles were placed in bucket C and 7 negative particles were also placed in bucket C, 5 positive and five negatives were attracted to each other and 2 negative charges are left.

The pupils may do many other activities such as the following and in each case ask, "What is the end result?"

ADD four positive charges to a bucket containing 8 negative charges

ADD five positive charges to a bucket containing 5 negative charges

ADD eight negative charges to a bucket containing 3 positive charges

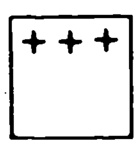
When the teacher is sure that the pupils understand the idea, he may then proceed to the recording of the data in number sentences.

For example, the recording of the illustration given at the beginning of this activity is $^-7 + {}^+5 = {}^-2$.

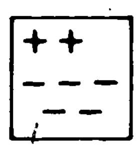
obj. 12c

51. The idea of charged particles as in the addition of integers may be applied in a model for subtraction.

A bucket contains 3 positive charges.



What would have to be added to it to make the bucket have a charge of negative 2?



Obviously 5 negative charges are needed as it takes 3 negatives to neutralize the 3 positives and 2 more negatives to make the bucket have a negative charge of 2.

The addition sentence then would be $^+3 + \square = {}^-2$. The replacement for $\square$ was found to be $^-5$; therefore, $^+3 + {}^-5 = {}^-2$

The related subtraction number sentence would be

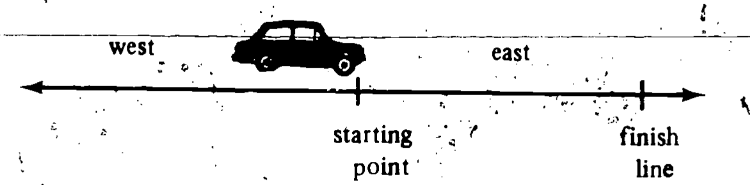
$$^-2 - {}^+3 = {}^-5$$

obj. 12c

52. Model cars on a race track or pictures of cars on a track may be used. The starting point on the track will be 0. The position of the car facing east toward the finish line will be *positive* and that of the car facing west in an opposite direction from finish line will be *negative*. Also the forward gear will be positive and the reverse gear will be negative.

Illustrate the following situations.

a. A car is headed toward the finish line and is in a forward gear. In which direction will it move?



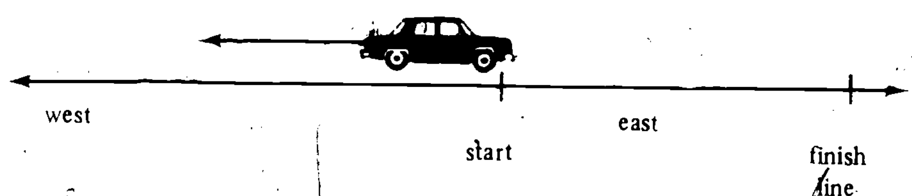
Explanation

Car headed east (+)

Forward gear (+)
Car will travel east +

(+,+) associates with → +

b. A car is headed toward the finish line and is in reverse gear. In which direction will it travel?

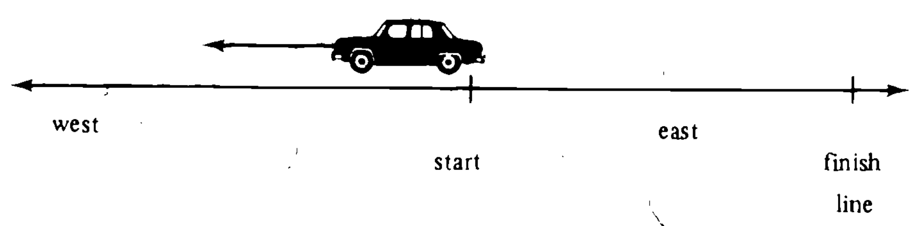


Explanation

Car headed east (+)
Reverse gear (–)
Car will travel west –

(+,–) associates with → –

c. A car is headed in the opposite direction from the finish line and is in forward gear. In which direction will it move?

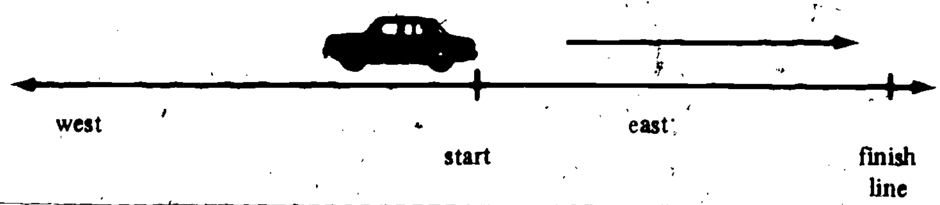


Explanation

Car headed west (–)
Forward gear (+)
Car will travel west –

(–,+) associates with → –

d. A car is headed in an opposite direction from the finish line and is in reverse gear. In which direction will it move?



Explanation

Car headed west (–)
Reverse gear (–)
Car will travel east +

(–,–) ——associates with——→ +

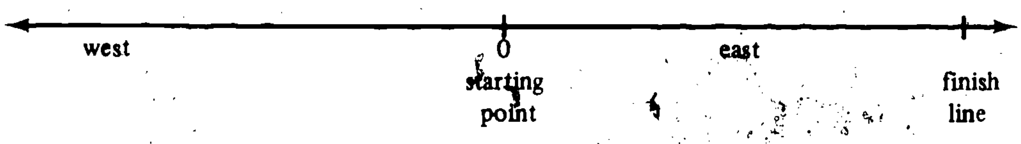
The teacher may work with these models until the pupils have a thorough understanding. Then he may explain that an operation such as (+,+) *associates with* +, relates to multiplication and is recorded.

a. (+) X (+) = (+)
b. (+) X (-) = (-)
c. (-) X (+) = (-)
d. (-) X (-) = (+)

Note: These recordings are for the above illustrations and are in the same order.

obj. 12c

53. A variation of the previous activity can be made by putting a paper track on the floor and by using pupils to move along the track in place of cars.



The pupil will face toward the finish line or away from the finish line and will move forward or backward as directed. For example, the pupil is directed to face the finish line and walk backward. Now the questions asked are wha[illegible]ne and what is the result using + and – symbols.

The pupils' response should be a[illegible] was facing the finish line (east) and that is positive (+). I walked backward and that is neg[illegible]). The result was I was walking west and that is negative (–). Therefore, a positive and a negative associate with a negative." (+,–) ***associates with*** → –

Examples showing all possible combinations should be used many times before explaining that this operation relates to multiplication and before recording with symbols.

Integers: Division

obj. 12c

54. The interpretation of division as the inverse of multiplication can be initiated by exercises in which the student finds the missing factor. The following are some examples.

a. $\square \times {}^{-}2 = {}^{-}10$, so ${}^{-}10 \div {}^{-}2 = \square$
b. ${}^{-}3 \times \square = {}^{+}6$, so ${}^{+}6 \div {}^{-}3 = \square$
c. ${}^{-}4 \times \square = {}^{-}8$, so ${}^{-}8 \div {}^{-}4 = \square$

d. $\square \times {}^{-}5 = {}^{+}25$, so ${}^{+}25 \div {}^{-}5 = \square$

e. $\square \times {}^{-}1 = {}^{-}3$, so ${}^{-}3 \div {}^{-}1 = \square$

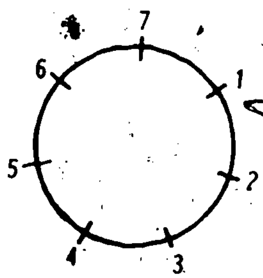
f. ${}^{+}7 \times \square = {}^{-}49$, so ${}^{-}49 \div {}^{+}7 = \square$

g. ${}^{+}5 \times \square = {}^{-}15$, so ${}^{-}15 \div {}^{+}5 = \square$

As can be seen ${}^{+}8 \div {}^{-}2 = {}^{-}4$ *because* ${}^{-}4 \times {}^{-}2 = {}^{+}8$, and division is the inverse of multiplication.

Supplementary Activities for This Strand

55. The teacher may draw on the chalk board, make ditto copies or make a transparency of the circular number line.



Say to pupils, "Look at the number line." "Does it resemble the clock?" The teacher may say, "Beginning at the point marked 2 and moving in the direction the hands move of a clock, move two units. Where are you?" Begin with six and make two moves. Where are you?" "Begin with 5, make four moves. Where are you?"

$1 + 4 = 5$

$6 + 2 = 1$

$5 + 4 = 4$

Complete the following.

$4 + 5 =$

$2 + 5 =$

$5 + 5 =$

$6 + 3 =$

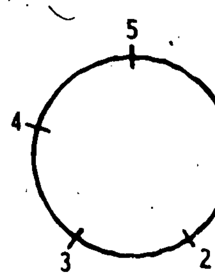
$5 + 4 =$

$5 + 2 =$

$3 + 6 =$

Multiplication on the circular number line can be thought of as repeated addition. The teacher may say, "Begin at 12, make 3 moves 4 units each. What is your answer?" After a number of similar questions ask, "What are some similarities between the procedure of multiplication here and that on a straight number line?" The pupils may be asked to develop a multiplication table for the 12-clock arithmetic.

56. You can think of subtraction on the clock as going in the opposite direction from addition. Ask, "Begin at two, make five moves in counterclockwise direction. Where are you?" $2 - 5 = 4$. What are some similarities between these procedures and the procedures for subtraction on a regular number line?



Using a 5-clock the teacher may ask the children to complete the following.

5 – 3 =

4 – 1 =

2 – 4 =

obj. 16

57. Use an 8-clock and have the children notice that a mathematical system can be built. Complete tables for other operations such as the one for addition.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

Have the children to investigate the system to answer the questions about the system.

a. Is this system closed under addition? How can you tell?

b. Is addition commutative in this system? How do you know?

c. Is addition associative in this system? How do you know?

d. Is there an identity element? If, yes, what is it?

e. If there is an identity element, does each element have an additive inverse?

obj. 16

58. For discussion of properties of operations present the following to the pupils.

$7 + 4 \stackrel{?}{=} 4 + 7$

$3 - 2 \stackrel{?}{=} 2 - 3$

$\frac{1}{2} + \frac{1}{4} \stackrel{?}{=} \frac{1}{4} + \frac{1}{2}$

$\frac{1}{8} - \frac{1}{12} \stackrel{?}{=} \frac{1}{12} - \frac{1}{8}$

$3 \times 2 \stackrel{?}{=} 2 \times 3$

$\frac{1}{3} \times \frac{1}{4} \stackrel{?}{=} \frac{1}{4} \times \frac{1}{3}$

${}^{-}2 + {}^{+}3 \stackrel{?}{=} {}^{+}3 + {}^{-}2$

${}^{-}7 - {}^{-}3 \stackrel{?}{=} {}^{-}3 - {}^{-}7$

Ask pupils to compare the results of the examples and to determine properties involved. Extend this to include not only commutative but other properties.

OPERATIONS, THEIR PROPERTIES AND NUMBER THEORY

OBJECTIVES

The pupil should be able to do the following.

1. Select appropriate operations on whole numbers for a given physical situation or illustrate a given operation by a physical situation.
2. Identify odd and even numbers
3. Discuss the properties of odd and even numbers
4. Identify prime and composite numbers
5. Give the prime factorization of any whole number
6. Name pairs of two whole numbers which are relatively prime
7. Find the greatest common factor of a set of numbers
8. Find the least common multiple for a set of numbers
9. Identify number patterns
10. Complete number patterns
11. Select the appropriate operation, addition or subtraction, on rational numbers for a given physical situation or illustrate a given operation by a physical situation
12. Use closure and the commutative, associative, distributive, identity, multiplicative and additive inverse properties to help him in his computation
    - (a) on whole numbers
    - (b) on rational numbers
    - (c) on integers
    - (d) using computation
    - (e) in formally describing the structure of a number system
13. Use the cancellation property to facilitate the solution of equations
14. Demonstrate immediate verbal recall of any basic facts
15. (a) Find the sum, product difference and quotient for any two whole numbers, that is if a difference or a quotient exists

    (b) Find the sum, product, difference and quotient for any two rational numbers, that is if a quotient exists
16. Use the closure, commutative, associative, distributive, multiplicative and additive identity, multiplicative and additive inverse properties to describe the structure of the number systems
    - (a) whole numbers
    - (b) rational numbers
    - (c) integers
17. Compare the structional likenesses and differences of the number systems
    - (a) whole numbers
    - (b) rational numbers
    - (c) integers

RELATIONS AND FUNCTIONS

INTRODUCTION

Relations, the idea of pairing or corresponding in a certain order, is basic to all mathematics. Beginning even at the preschool level, the pupil can learn through experience to recognize relations, to use them in formulating his own ideas and to show in communicating to others that he is developing intuitively a pattern of organized thinking in nonnumerical situations. By using relational thought patterns in his early experiences, he establishes readiness for extending these concepts in mathematical situations as he meets in his development. Therefore, the teacher should see that from the beginning a foundation for correct concepts is laid so that unlearning will not be necessary later.

Pupils encounter many nonnumerical relations; many can be found in stories for primary children. Some of these nonnumerical relations, such as *belongs to, is brother of*, and *is in the same house as* should be used before numerical relations to illustrate the meaning of relations. These relations can also be used to lead into numerical relations since they can be examples of correspondences of *one-to-one, many-to-one, one-to-many*, or *many-to-many*. Such correspondences are basic to the idea of number, to the relations *equal to, less than*, and *more than*, and to the operations with numbers.

There are several special kinds of relations with special names. One of these, called an equivalence relation, is associated with the process of classification. Classification is the process of partitioning a set of elements into different subsets in which no element can belong to more than one subset. This, too, can be introduced through nonnumerical situations. For example, a set of blocks can be separated into classes on the basis of color provided the colors are distinct. Or, a set of coins can be partitioned into subsets according to value. Such subsets, or classes, are called equivalence classes, and the relation exemplified by their membership, *same color as* or *same value as*, is called an equivalence relation. When school children are classified by grade in school, if no pupil can be in more than one grade, the different grades represent equivalence classes, and the equivalence relation is *is in the same grades as.* Equivalence relations are very important in mathematics. The most familiar is the one called *is equal to*, but many others are encountered as the pupil progresses through mathematics.

Another special kind of relation is that known as a function, or mapping. Although the concept of a function is one of the unifying themes of mathematics, it is unwise to introduce pupils to the concept by giving a formal definition. If the pupils have sufficient practice in pairing elements of one set with elements of another while studying relations in general, those having the special property required of functions will not be difficult to identify. It is for this reason that early activities on relations in the guide include the suggestion that pupils write out the ordered pairs associated with a relation. It is also suggested that pupils make graphs of relations. As the pupils observe many different kinds of graphs, those graphs characterizing functions will stand out in sharper focus.

Also important in mathematics are the special relations called order relations, such as *more than* and *less than.* These are used when such concepts as *heavier than, longer than, darker than* or *thinner than* are being considered. Measurement such as that of time, capacity and length consists of ordering the quality to be measured and then assigning numbers to correspond to that ordering. Thus the numerical order relation makes precise the intuitive one.

The activities in this strand include suggestions for introducing pupils to relations in general and to the special relations discussed above. As with other strands, the teacher will need to select those which are appropriate for his class and supplement them as necessary. It should be re-emphasized, however, that familiarity with relations in general should precede formal work with special relations.

Mathematics can be viewed as an entity of systems, each consisting of the following components—sets of elements, or basic units such as whole numbers, rational numbers, or points; relations or comparisons of these elements, such as *equal to, greater than* or *congruent to;* and operations such as multiplication or set union. Therefore, throughout each strand in elementary mathematics, recognizing and using relations constitutes a basic activity in which the pupil must engage in order to understand the concepts included in that strand.

RELATIONS AND FUNCTIONS

Objectives Keyed to Activities

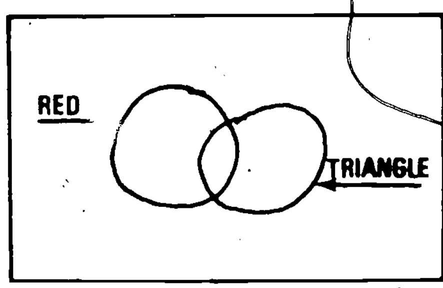
ACTIVITIES

obj. 1

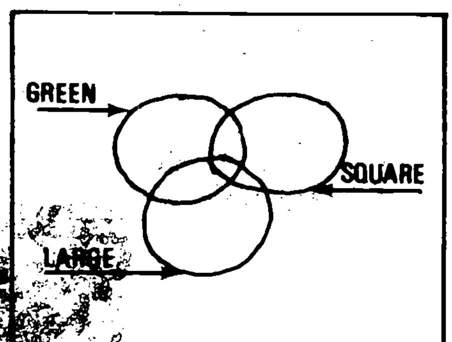
1. a. From three colors of construction paper cut one each of the following shapes – circular, triangular and square. Ask pupils to sort the discs according to shape or color and enclose each subset within a circle of yarn.

b. Extend the activity above by also making discs of two different sizes.

c. Place two loops of yarn on a table and label them as shown. Have a student or group of students place the discs in the loops according to the labels. Turn the labels over and ask other pupils who did not observe this activity to name a common characteristic of the pieces inside the loops.



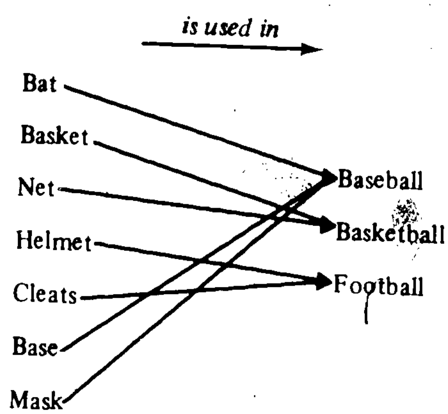
d. Place three loops on the table as shown below and use the same procedure as with the two loops.



e. Increase the number of properties for the pupils to consider by using blocks of different sizes, shapes, colors and thicknesses.

obj. 1

2. a. List items of equipment used in certain sports and record the relation *is used in the same sport as.*



b. Then partition the same set of items using the relation *is in the same category as.*

Baseball	Basketball	Football
Bat		
Base	Basket	Helmet
Mask	Net	Cleats

Since this set can be partitioned into disjoint sets, the relation *is in the same category as*, is an equivalence relation and each subset forms an equivalence class. Lead the pupils to see that equivalence relations have certain properties. The reflexive and symmetric properties may be too abstract to discuss at this level, but the transitive property should be discussed. The three properties necessary for equivalence and examples of each are given below.

Reflexive – The bat is in the same category as itself.

Symmetric – If the bat is in the same category as the base, then the base is in the same category as the bat.

Transitive – If the bat is in the same category as the base and the base is in the same category as the mask, then the bat is in the same category as the mask.

obj. 5,11a, 12 3. a. An example of a nonnumerical relation is, *is the child of.* This relation applied to the Jones family consisting of Mother, Father, Tom, Mary and Susan gives six pairs.

Tom, Mother (i. e., Tom *is the child of* Mother).
Tom, Father
Mary, Mother
Mary, Father
Susan, Mother
Susan, Father

In set notation the relation is written as follows.

Is a child of = {(Tom, Father), (Tom, Mother), (Mary, Father), (Mary, Mother), (Susan, Father), (Susan, Mother)}

b. Have the pupils write in set notation the relation *is the father of* for the Jones family.

c. Have the pupils write in set notation the relation *is the brother of* for the Jones family.

d. Use other examples to develop the understanding of ordered pairs. Have pupils list the ordered pairs of the following relations – *was born before* defined on {George Washington, Abraham Lincoln, Christopher Columbus, Neil Armstrong}; *is north of* defined on {Atlanta, Miami, Chicago, Knoxville}; *is greater than* defined on {8, 9, 10, 11}.

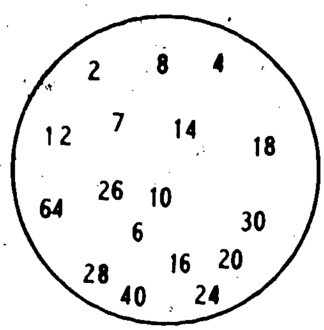
obj. 6

4. a. Have pupils find the rule which the arrow represents in patterns such as these.

$1 \longrightarrow 2 \longrightarrow 4 \longrightarrow 8 \longrightarrow 16$	(Here $\longrightarrow$ represents $\times$ *2*)
$3 \longrightarrow 7 \longrightarrow 15 \longrightarrow 31$	(Here $\longrightarrow$ represents *double and add 1*)
$1 \longrightarrow 4 \longrightarrow 16 \longrightarrow 64$	(Here $\longrightarrow$ represents ____________)
$5 \longrightarrow 12 \longrightarrow 26 \longrightarrow 54$	(Here $\longrightarrow$ represents______________)

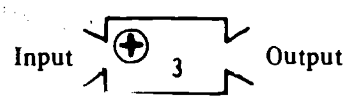
b. Have pupils make up some patterns of the type given in a. and challenge each other.

c. Find as many relationships as you can between some of these numbers. Use arrows to show the relations, and use different colors for different relations.

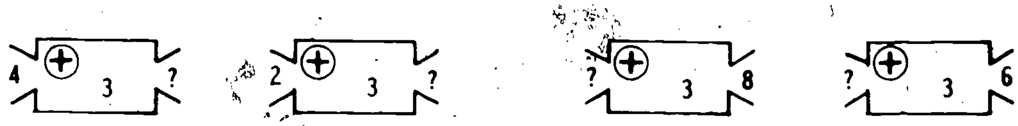


obj. 4,5,6

5. Extend the use of machines as described in the lower grades. Give the following drawing of an *add 3* machine. That is, every number put into the machine has 3 added to it.

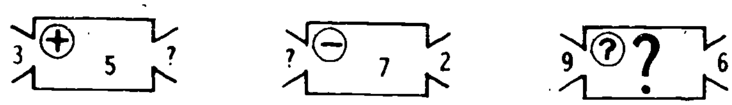


Give the following examples and ask what number would go into or come out of each machine.



Give other machines such as *add 5, add 7, subtract 4, multiply 6,* etc. After the pupils are familiar with the machines give them some drawings as shown below and ask what each machine does.

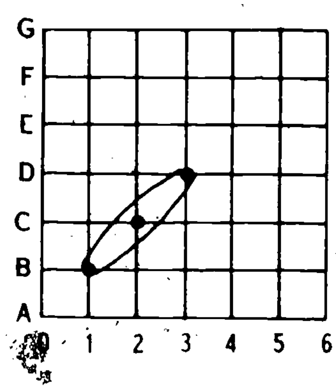
Then, give machines with various missing facts, such as the following



Have pupils make up machines to challenge each other.

obj. 9,14

6. The game of battleship is useful in introducing graphing on a coordinate plane. Two players or two competing groups may play. Each group places a battleship touching or including three points on the same line of a Cartesian plane as shown. With younger pupils label one of the axes with letters thus using ordered pairs of numerals and letters. Later use ordered pairs of numerals.



Each group of players keeps two graphs, one (A) with its battleship hidden from their opponents; the other (B) to record and keep track of the shots made on the opponents battleship. The groups take turns shooting torpedoes at the opponent's battleship by naming three points on the plane, such as (1,B), (4,E), (5,C). Each group should mark the shots on its graph as they are announced. In like manner each group should mark its own shots on its (B) graph. As the shots are given, the opponent must tell how many times the ship was hit but not which shots hit the ship. Play continues until one battleship has been sunk by being hit three times.

A more difficult game may be devised by increasing the number of ships and by varying their sizes, such as including a carrier (5 points), a battleship (4 points), a destroyer (3 points) and a submarine (2 points). Each turn would consist of four shots, and as the shots are announced the opponent tells which type of ship was hit. As the group calls the shots, they should mark them on a blank graph, for a miss they could write 0 and for a hit a number indicating the number of points for the type ship that was hit. Play continues until all of the ships of one group have been sunk. If the group who had the first turn succeeds first in sinking the opponent's ships, then the other group is entitled to one more turn to either tie or win the game.

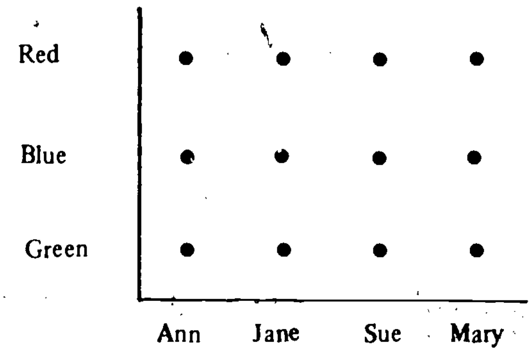
Teachers may find assistance in planning activities of this type in the books listed in the section on media.

obj. 1,9

7. a. Have two girls and four boys come to the front of the class. Ask the class how many different dancing couples can be identified by using a different girl and boy for each couple named. The couples should be identified and their names recorded. After all pairings have been made, the total number should be determined. The set of all couples named or the set of all ordered pairs made by pairing each member of the first set with each member of the second set forms the Cartesian product of the two sets.

A number of other situations of pairing the members of two sets should be used to help children understand how Cartesian products are formed. For example, form a Cartesian product using a set of three boys and a set of motorized vehicles.

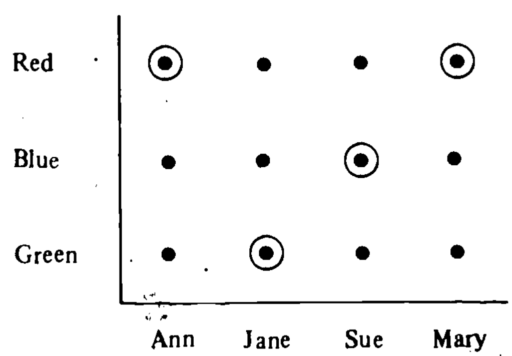
b.



The accompanying figure is a graph of the following possible ordered pairs of the relation *is wearing the color of*. The pairs are represented by lattice points in the graph.

Is wearing the color of = {(Ann, red), (Jane, red),
(Ann, blue), (Jane, blue),
(Ann, green), (Jane, green),
(Sue, red), (Mary, red),
(Sue, blue), (Mary, blue),
(Sue, green), (Mary, green)}

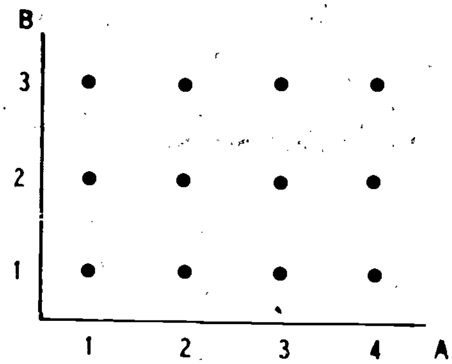
In the second figures the points which are circled make a graph showing what colors the girls are actually wearing.



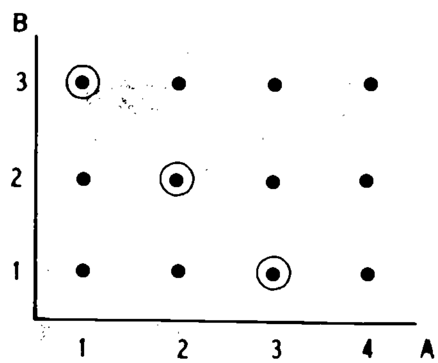
State the ordered pairs of the points which are circled. Are any of the girls wearing the same color?

c. Have the pupils pair the members of the set $A = \{1, 2, 3, 4\}$ with those of $B = \{1, 2, 3\}$ to form the Cartesian set called $A \times B$, read "A cross B," and then draw the set of lattice points for $A \times B$.

$A \times B = \{(1,1), (2,1), (3,1), (4,1), (1,2), (2,2), (3,2), (4,2), (1,3), (2,3), (3,3), (4,3)\}$



The pupils should have practice graphing Cartesian sets and also in selecting and graphing solution sets for open sentences. For example, the subset of $A \times B$ which is the solution set for $\square + \triangle = 4$ is $\{(3,1), (2,2), (1,3)\}$ and the graph is shown. Notice that the solution set is embedded in the Cartesian product.



Other examples of open sentences should be given such as the following.

$\triangle > \square$

$\triangle = \square + 1$

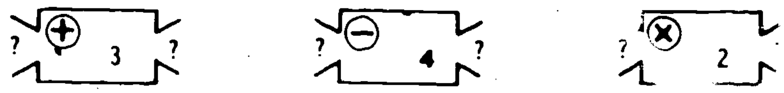
Other examples of Cartesian sets are as follows.

$C \times D$ where $C = \{0, 1, 2\}$ $\{D = 1, 2, 3, 4\}$

$E \times F$ where $E = F = \{1, 2, 3, 4, 5\}$ and the like.

obj. 4,5, 6,11

8. After the pupils are able to find missing parts of machines as in number 5, ask them to find pairs of numbers for machines such as those illustrated.



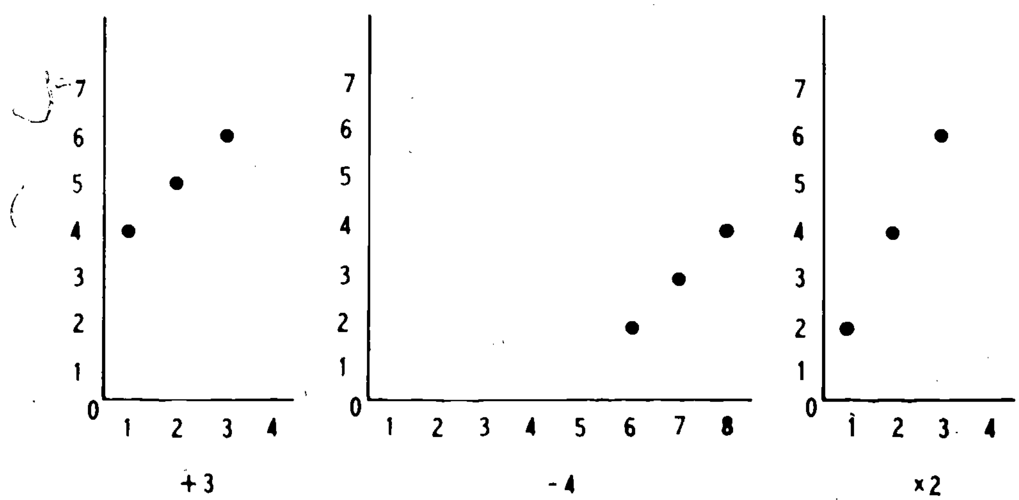
Have pupils record their findings in tables such as the following.

+ 3	
1	
2	
3	

- 4	
8	
7	
6	

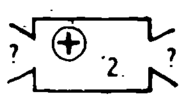
x 2	
1	
2	
3	

Graphs of some of these pairs are as follows.

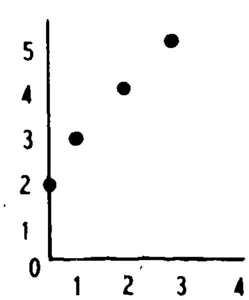


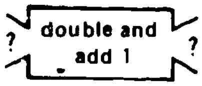
Give students various types of machines and ask for tables and graphs.

Examples

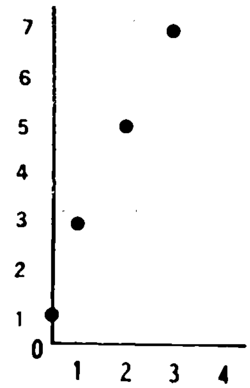


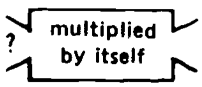
+ 2	
0	
1	
2	
3	



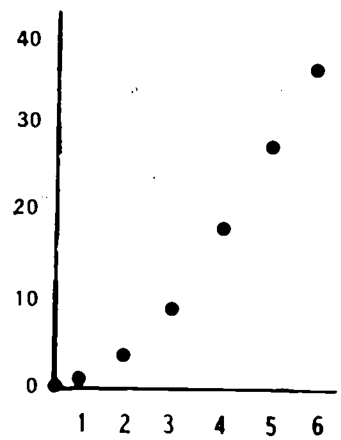


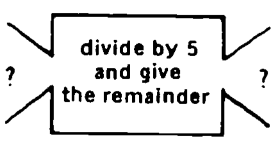
double and add 1	
0	
1	
2	
3	



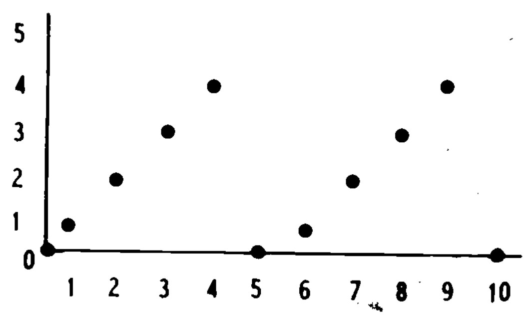


multiplied by itself	
0	
1	
2	
3	
4	
5	
6	

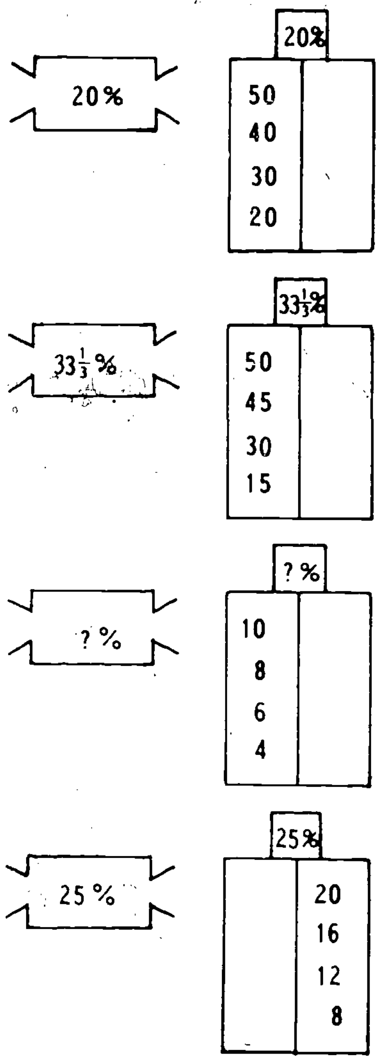




divide by 5 and give the remainder	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

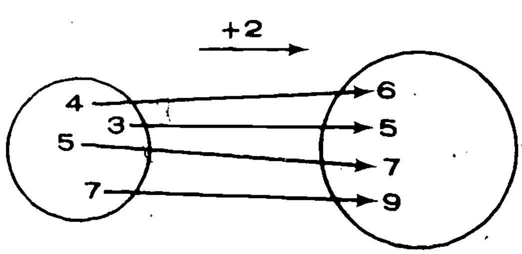


Extend the use of machines to the study of fractions, decimals, per cents and integers, for example using per cent.



9. a. Ask each child to hold up one pencil as an illustration of one-to-one correspondence of pupils to pencils.

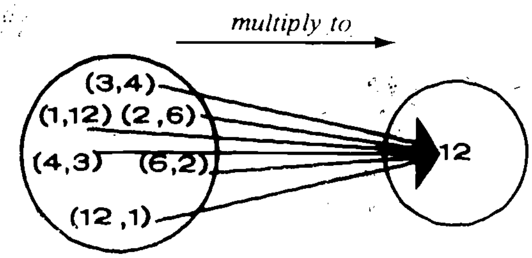
b. Show an example of one-to-one correspondence using numbers.



c. Have pupils suggest other examples, both numerical and nonnumerical, of one-to-one correspondences. Make diagrams similar to the one in 9b to reinforce the idea of pairing.

obj. 2c

10. a. Show an example of many-to-one correspondence using numbers. Ask pupils to give examples with the relation *multiply to.*

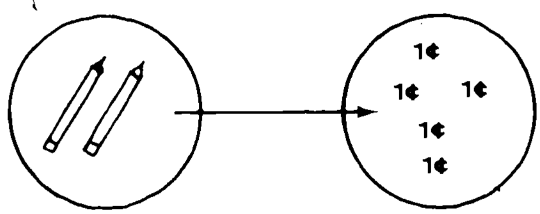


b. After other examples are discussed, ask if any ordered pairs which *multiply to* 12 also n *to* any other number? Of course, the answer is no, and this is true for each of the other pairs in any other set under the relation *multiply to.* Therefore, these sets are disjoint, t equivalence classes in the same sense as in activity 2.

c. Relations arising in earlier activities might be investigated to see which are many-to-on spondences.

obj. 2c,2d, 3

11. a. Map the many-to-many correspondence of the relation *2 pencils cost 5 cents* and ask p suggest other examples.



b. Use a correspondence such as the following example.

△ △ ⟶ □ □ □

△ △ ⟶ □ □ □

△ △ ⟶ □ □ □

△ △ ⟶ □ □ □

Record the relations as follows.

two triangles match ______ squares

four triangles match ______ squares

six triangles match ______ squares

eight triangles match ______ squares

Record as ordered pairs by completing the following.

(2, __), (4, __), (__ , 9), (__ , __)

Record as ratios by completing the following.

$\frac{2}{3}$, $\frac{4}{?}$, $\frac{?}{9}$, $\frac{?}{?}$

Use other combinations of triangles and squares or similar physical representation to strengthen understanding of ratios.

c. In the many-to-many correspondences in parts a. and b. there is an infinite set of representations for each correspondence. In example a., the correspondence of 2 pencils to 5 cents could also be represented as 4 pencils for 10 cents, or 6 pencils for 15 cents, giving the following ratio form.

$$\left\{ \frac{2}{5}, \frac{4}{10}, \frac{6}{15}, \frac{8}{20}, \ldots, \frac{2n}{5n}, \ldots \right\}$$

This infinite set is called an equivalence class, and in the general term, n stands for any counting number. The equivalence class generated in b. can be written as follows.

$$\left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{4}{12}, \ldots, \frac{2n}{3n}, \ldots \right\}$$

Ratios do not come exclusively from many-to-many correspondences, but can come from one-to-many or many-to-one, giving the following examples of equivalence classes.

$$\left\{ \frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \ldots, \frac{n}{4n}, \ldots \right\}$$

$$\left\{ \frac{5}{1}, \frac{10}{2}, \frac{15}{3}, \frac{20}{4}, \ldots, \frac{5n}{n}, \ldots \right\}$$

$$\left\{ \frac{10}{3}, \frac{20}{6}, \frac{30}{9}, \ldots, \frac{10n}{3n}, \ldots \right\}$$

12. a. Equivalence classes generated in activity 11 should be used to develop proportion. A proportion is any pair of elements in one equivalence class of ratios. $\frac{1}{2} = \frac{3}{6}$ is a proportion.

To use the equivalence classes, proceed as follows.

Problem $\frac{2}{3} = \frac{8}{?}$

Procedures

(1) $\frac{2}{3} = \frac{2n}{3n}$ This sentence is true since the ratios belong to the same equivalence class.

(2) $\frac{2n}{3n} = \frac{8}{?}$ The problem can be restated using the transitive property of equality.

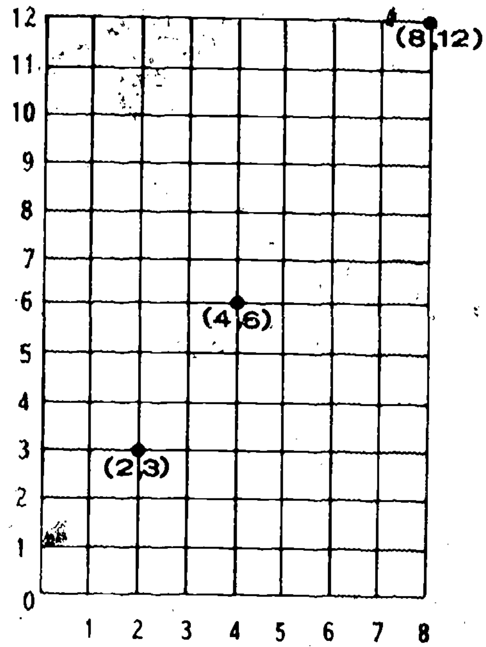
(3) 2 $\boxed{\times N}$ 8 By using a machine, determine that N = 4.

(4) 3 $\boxed{\times 4}$? By a second determine that " ? " is 12.

That is $\frac{2}{3} = \frac{8}{12}$.

Exercises in proportion may be found in textbooks. Applications of ratio and proportion may be found in the study of similar figures in the strand Geometry.

b. Equivalence classes of ratios can be shown graphically. Use different colors for the different classes of ratios. The pupils should be made aware that *all* points of an equivalence class of ratios lie on one line. The infinite set $\left\{\frac{2}{3}, \frac{4}{6}, \frac{8}{12}, \ldots, \frac{2n}{3n} \ldots\right\}$ with n a counting number can be shown as follows.



obj. 3 13. For applications of proportion, have the pupils measure distances on maps in an atlas or on road maps and determine approximate distances in miles from one location to another.

obj. 3,4 14. A ratio which has 100 for the second component may be represented with the symbol for per cent such as 5% for $\frac{5}{100}$. Since any ratio can represent its equivalence class, the pupils should have experience generating classes of ratios which include some with 100 as a second component before doing any computation with percents.

$$\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots, \frac{50}{100}, \ldots, \frac{n}{2n}\right\}$$

$$\left\{\frac{3}{10}, \frac{6}{20}, \frac{9}{30}, \ldots, \frac{\quad}{100}, \ldots, \frac{3n}{10n}\right\}$$

$$\left\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \ldots, \frac{25}{100}, \ldots, \frac{n}{4n}\right\}$$

$$\left\{\frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \ldots, \frac{20}{100}, \ldots, \frac{n}{5n}\right\}$$

By inspection, $\frac{2}{4}$ and $\frac{3}{6}$ both belong to the $\frac{50}{100}$ or the 50% class, so 2 is 50% of 4 and 3 is 50% of 6. Similarly, $\frac{9}{30}$ belongs to the 30% class, so 9 is 30% of 30.

When computation with percent is introduced, choose ratios so that the second component is a factor of 100.

Example

since $\frac{6}{20} = \frac{30}{100}$

$6 = \frac{30}{100} \times 20$

6 is 30% of 20

since $\frac{1}{4} = \frac{25}{100}$

Therefore, 1 is 25% of 4.

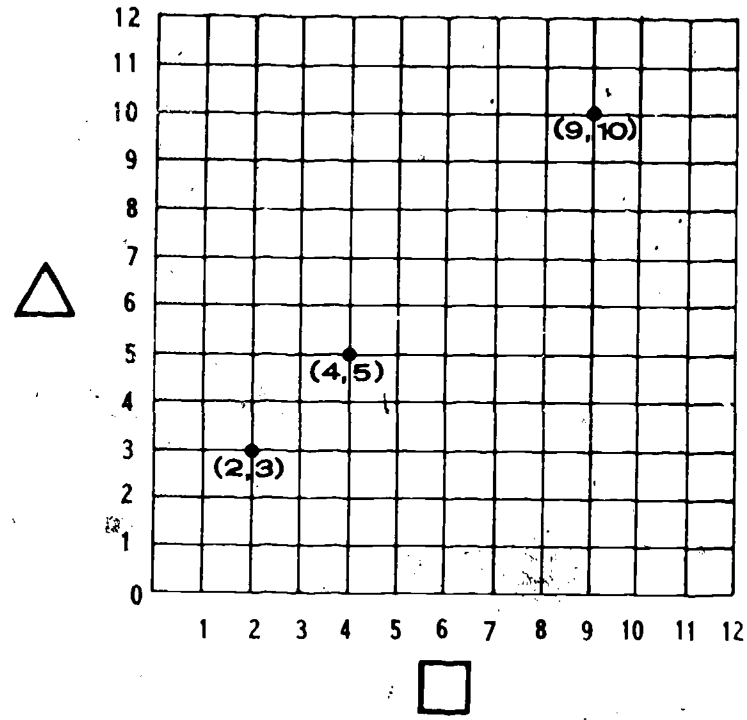
obj. 9,11

15. a. Ask the pupils to consider the open sentence $\Delta + 3 = \square + 4$ with the truth set defined on $\{0, 1, 2, 3, \ldots 12\}$. Some ordered pairs of $(\square, \Delta)$ that make the statement true are (2,3), (9,10), (4,5). Have the pupils to record the ordered pairs by completing the table shown. In problems of this type if the first elements of the pairs are written in sequential order in a column of the table, a pattern emerges.

$\square + 4 = \Delta + 3$	
$\square$	Δ
0	
1	
2	3
3	
4	5
5	
6	
7	
8	
9	10
10	

Have the pupils complete the graph of the sentence as defined on the given set.

$\square + 4 = \triangle + 3$



b. Have the pupils make a table and plot the graph of ordered pairs of the solution set of $\triangle = \square + 2$ defined on $\{0, 1, 2, 3, \ldots 10\}$.

c. Have the pupils make tables and plot the graphs of ordered pairs of the solution sets of equations such as $\triangle = 2 \times \square + 1$ defined on $\{0, 1, 2, 3, \ldots 6\}$.

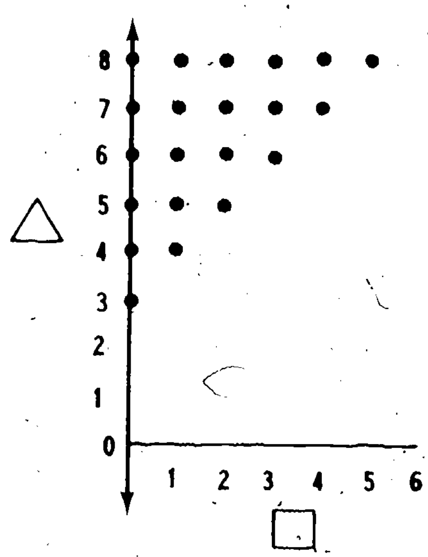
d. Extend these activities including equations defined on the set of whole numbers and the set of integers.

obj. 14

16. Have the pupils plot graphs of inequality relations defined on the set of whole numbers and the set of integers.

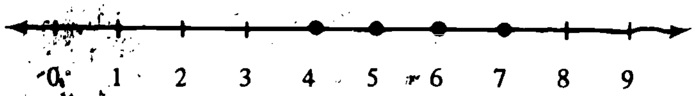
Example

Graph the truth set of $\triangle > \square + 2$ defined on the set of whole numbers.



obj. 13

17. a. A number line may be used to represent the *truth set* for an inequality on the set of whole numbers.



$3 < \square \leq 7.$

Have the pupils graph the solution sets of the following inequalities on number lines.

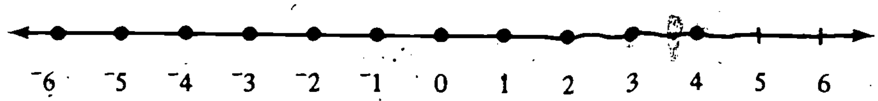
$\square > 4.$

$1 < \Delta < 5.$

$2 \leq \square < 8.$

b. A number line may also be used to graph the *truth set* for an inequality on the set of integers.

$\square < 5$



obj. 6,10

18. a. In introducing the study of functions, use nonnumerical relations such as the ones suggested in this activity. Functions are special relations. A relation in which each first element is paired with only one member of the set of second elements is a function. Functions (or mappings) can be correspondences that are one-to-one or many-to-one.

Have the students write the relation *is the son of* defined on a set of men {Mr. Jones, Mr. Brown, Mr. Smith} and a set of boys {John, Tom, Sam, Bill}

The relation would be – *is the son of* = {(Sam, Mr. Smith), (Bill, Mr. Jones), (Tom, Mr. Smith), (John, Mr. Brown)}.

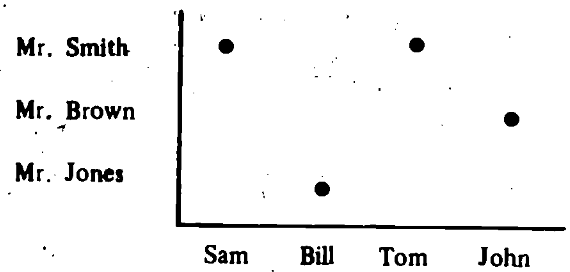
Since each boy has only one father, the name of only one man can be paired with a name of a boy in that set. Therefore, the relation *is the son of* is a function.

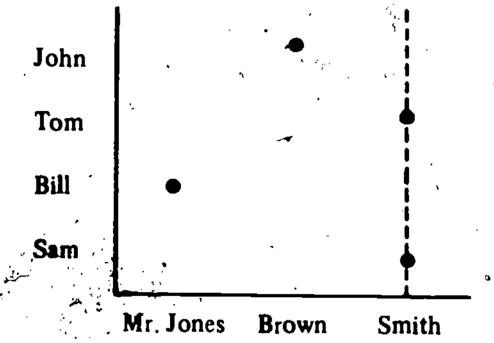
The relation *is the father of* is not a function. Using the same sets of men and boys given above, the relation would be as follows.

is the father of = {(Mr. Smith, Sam), (Mr. Jones, Bill), (Mr. Smith, Tom), (Mr. Brown, John)}.

Since a father (Mr. Smith) can have more than one son, the relation *is the father of* is not a function.

The graph of a relation can readily show if the relation is a function. If any line parallel to the vertical axis intersects the graph (if at all) in only one point, then the relation is a function. The graphs of the two relations stated above are given here as illustrations. In the graph of the second relation the vertical dotted line contains two points of the graph; therefore, this relation is not a function. No vertical line contains two points of the first relation; therefore, this relation is a function.





b. Ask the pupils to select from the following sets of ordered pairs those which are functions.

(1) { (John, Mary), (Tom, Agnes), (John, Susan), (Bill, Jane) }

(2) { (cat, kitten), (dog, puppy), (cow, calf), (hen, chick) }

(3) { (baseball, bat), (golf, club), (tennis, racket) }

(4) { (fruit, orange), (nut, pecan), (vegetable, corn), (fruit, peach) }

The pupils will find that (2) and (3) are functions, but (1) and (4) are not functions because *John* is the first member in two pairs and *fruit* is the first member in two pairs.

c. The machines in activities 7 and 13 are sometimes called function machines. Have the pupils determine why these machines can be so named. Have them pay particular attention to the test for functions using graphs.

d. More capable pupils can be challenged to *find the rule* from tables of data. Below are some examples. The problem for the pupil is to determine how to get △ given the value for □ in each case.

□	△
0	1
1	2
2	3
3	4

□	△
0	1
1	3
2	5
3	7

□	△
0	2
1	5
2	8
3	11

□	△
0	1
1	2
2	5
3	8

It is sometimes helpful to have activities which are extra for experts on separate 3" X 5" cards so the pupils can work on these independently.

obj. 5, 11

19. a. On a grid or pegboard, have pupils outline several rectangular regions having a given area (12 square units, for example), but having different dimensions. Compute the perimeter of each of these rectangles.

b. Using the data from part a., have the pupils make graphs, plotting the width of the rectangle on the horizontal axis, the perimeter on the vertical axis. Some questions which can be raised about the graphs are as follows.

- Can the width of the rectangle be smaller than 1?
- Can the width of the rectangle be larger than 3 (assuming the given area is 12 as suggested)?
- Can the rectangle have a perimeter of 5?
- Can the rectangle have a perimeter of 49?
- What is the smallest perimeter this rectangle can have?
- What is the largest perimeter this rectangle can have?

c. As a result of the questions, the pupils may want to add some more points to their graphs so that the pattern becomes more distinctive.

d. A variation of the activity in parts a. through c. is to assign a given perimeter to the rectangle (12 units for example), and investigate the various areas enclosed by rectangles of given perimeter but different dimensions. Comparison of the graph to be made here and that in b. is instructive and to some pupils surprising.

obj. 7

20. a. Have the students solve by inspection equations such as those given below.

$\square + 5 = 11$ $\quad$ $\square - 2 = 8$ $\quad$ $\dfrac{\square}{5} = 6$ $\quad$ $\dfrac{42}{\square} = 7$

$7 - \square = 3$ $\quad$ $3 \times \square = 27$ $\quad$ $2 \times \square + 3 = 19$

$\square \times \square = 64$

Next, use letters to represent the missing numbers in equations which can be solved by inspection.

$x + 4 = 9$ $\quad$ $x^2 = 25$ $\quad$ $3y + 2 = 23$

$n - 8 = 7$ $\quad$ $\dfrac{a}{3} = 2$ $\quad$ $y = (3)(22)(0)$

[illegible] $y = 5$ $\quad$ $n = \dfrac{(2)(3)^2(5)}{(2)(3)^2(5)}$

$3 \times m = 18$ $\quad$ $\dfrac{28}{n} = 7$

Have the pupils write on slips of paper some equations to be solved by inspection. Exchange the papers and ask the pupils to solve the equations they receive.

b. Write on the chalkboard a few equations such as those given below. These should probably be presented one at a time and as puzzles for the pupils to figure out. Reasonably good students can succeed at this if given time to think and try out their hunches. The pupils should be permitted to solve the equation their own way and to discuss their methods before proceeding to the next example.

$\frac{x}{3} = 4$ (or $x \div 3 = 4$) $\qquad 3(t - 2) = 15$

$\frac{x+1}{3} = 4$ (or $(x + 1) \div 3 = 4$) $\qquad n^2 + 3 = 28$

$2(y + 1) = 20$ $\qquad \frac{n+3}{7} = 4$ (or $(n + 3) \div 7 = 4$)

$2(3y + 1) = 20$ $\qquad \frac{1}{2}x - 2 = 12$

$3x + 2(x + 4) = 28$

c. To introduce more formal methods for solving equations, the following pair can be used.

$2x + 6 = 8$

$6(x + 3) = 28$

If the pupils are proficient in finding solutions by inspection, these are easy. If not, suggest that they try some numbers. The conclusion is that both equations have the same solution, namely, 1. Equations which have the same solution set are called equivalent equations. Formal equation solving is a process of changing an equation into an equivalent one which is easier to solve. Since an equation is a sentence about equality of numbers, properties of numbers and operations and properties of equality are used in this process.

These properties of equality are shown below with a, b and c as numbers and three of arithmetic.

Reflexive: For each a, $a = a$.

Symmetric: For each a and b, if $a = b$, then $b = a$.

Transitive: For each a, b and c, if $a = b$ and $b = c$, then $a = c$.

The first two may seem trival but their importance will be realized as equalities are studied.

Other properties of equality are as follows;

Addition: For each a, b and c, if $a = b$, then $a + c = b + c$.

Multiplication: For each a, b and c, if $a = b$, then $ac = bc$.

Cancellation of addition: For each a, b and c, if $a + c = b + c$, then $a = b$.

Cancellation of multiplication: For each a, b, and c [illegible] 0, if $ac = bc$, then $a = b$.

Pupils who were successful at parts a. and b. will find that the following formal solutions essentially describe their methods.

Example 1 Solve $x + 4 = 9$

	$9 = 5$ [illegible]	
so	$x + 4 = 5 + 4$	transitive property
and	$x = 5$	cancellation of addition

Example 2	Solve $3 \cdot m = 18$	
	$18 = 3 \cdot 6$	
	so $3 \cdot m = 3 \cdot 6$	transitive property
	and $m = 6$	cancellation of multiplication
Example 3	Solve $2(y + 1) = 20$	
	$20 = 2 \cdot 10$	
	so $2(y + 1) = 2 \cdot 10$	transitive property
	and $y + 1 = 10$	cancellation of multiplication
	$10 = 9 + 1$	
	so $y + 1 = 9 + 1$	transitive property
	and $y = 9$	cancellation of addition

To explain the formal solution of an equation, such as $3(t - 2) = 15$, proceed as follows.

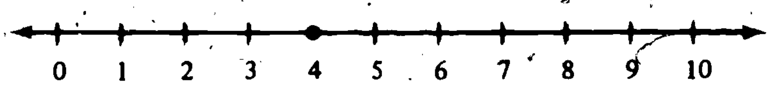
$3(t - 2) = 15$	
$15 = 3 \cdot 5$	
so $3(t - 2) = 3 \cdot 5$	transitive property
and $t - 2 = 5$	
then $(t - 2) + 2 = 5 + 2$	addition property
and $(t - 2) + 2 = t$	
so $t = 5 + 2$	transitive property
or $t = 7$	

The examples given demonstrate *how* properties of equality can be used to change an equation into an equivalent one; these examples do not explain *why* the solution is correct. A solution is correct only if it fits the condition of the problem. Thus, in example 1, 5 is the correct solution because $5 + 4 = 9$ and in example 3, 9 is the correct solution because $2(9 + 1) = 2 \cdot 10 = 20$.

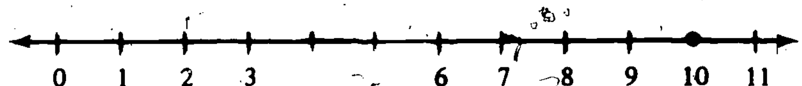
c. Have the pupils graph on number lines the solution of equations such as the following.

Answers

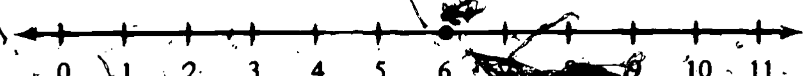
$n + 3 = 7$



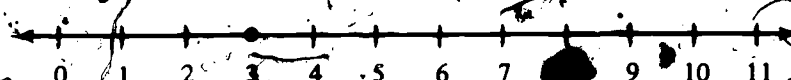
$5 + m = 15$



$8 - n = 2$



$2(y + 1) = 8$



Include equations whose solutions will be fractions and decimals.

RELATIONS AND FUNCTIONS

OBJECTIVES

The pupils should be able to do the following.

1. Classify elements of a set according to specified properties
2. Demonstrate correspondences (a) one-to-one, (b) one-to-many, (c) many-to-one and (d) many-to-many
3. Apply equivalence relations to elements such as fractions, ratios and geometric figures
4. Find the missing element of a pair when one member of the pair and the relation are given
5. Find some pairs of elements when a relation is given
6. Find the relation when a set of ordered pairs is given
7. Use the properties of equality
8. Graph equalities on the number line
9. Graph equalities on a coordinate plane
10. Determine is a given relation is a function
11. Record functions
    (a) set notation
    (b) sentences
    (c) formulas
    (d) mappings
    (e) tables
    (f) graphs
12. Order a set of elements according to a specified relation
13. Graph inequality relations on a number line
14. Graph inequality relations on a coordinate plane

GEOMETRY

INTRODUCTION

The topics begun in the primary grades should be further developed in the upper grades, taking into account that a particular class may need review of or even introduction to some of these topics. The pupil in the upper grades should be able to work at a more mature level, so that topics can be pursued at greater depth. However, even a seventh grade pupil will need some first-hand experience with rotations before he can recognize that some particular set of points is merely a rotation of another set of points.

There is more to the study of geometry than identification of shapes and measurement of figures. Recognition of relationships is very important. Consider the difficulty many pupils have in learning to use a protractor. Given the task of measuring the angles in the accompanying illustration a pupil may report 120° for (a) and 60° for (b).

(a) (b)

He is not likely to reject these answers unless he recognizes that the angles are congruent. That is, before a pupil can make use of a relationship, he must first of all recognize it. The intent of this guide is to provide opportunities for pupils to discover relationships and to recognize the conditions which give rise to those relationships.

For this reason, it is helpful if the study of geometry below the high school level is thought of as exploration of space. Exploration means searching, probing, making discoveries. The activities suggested in the following pages are intended to give a pupil the sort of first-hand experience which can properly be called exploration. By pulling, turning, sliding, folding he learns to predict the appearance of geometric figures under different conditions; by designing a pattern for a model he learns for himself what parts must be assembled and what arrangements of these parts will and will not work; by creating larger and smaller copies he discovers properties which are independent of size.

An explorer does not set out with a ready-made itinerary and an immutable timetable, but it is helpful if he can consult from time to time with someone who knows the terrain and can give him some advice. In exploring geometry, the teacher can be this consultant by suggesting other things to try, posing the right sort of question, and encouraging the pupil to find answers by experimenting with objects or representations of objects.

Finally, an explorer takes notes as he goes but writes the final report at the end of his journey. Thus, in geometry, formal definitions and precise statements of generalizations should be the culmination rather than the beginning of a journey through a topic. The teacher plays a vital role here, for the young explorer cannot discover names for what he has seen; these must be supplied by the teacher. The pupil pulls, turns, slides and folds pieces of balloons or pieces of paper with drawings on them and sees how drawings appear afterward. In discussing what he has found, he will find it helpful to have a single word to use instead of repeating the sequence "Pulling, turning, sliding or folding." The teacher can then supply the word *transformation* or *motion*. The pupil, however, does not need to know this word before he starts out; he can learn what the pictures look like without ever having heard the word. Likewise, in discussion it may be helpful to have a word to use instead

of "all these ways the drawing might look if I turned the paper," and again the teacher can supply the word *equivalent* for the set of drawings in question. The role of language in geometry should be to facilitate learning and communication; word study should not be an end in itself. A note of caution, however, is in order. Some words in geometry are also used in everyday life, but with a slightly different meaning. In these cases, it may be wise to call the pupil's attention to the familiar uses of the words and point out the restrictions relative to geometry. A good example is the word straight. "Going straight down the road" may not be straight in the geometric sense.

Since this volume of the guide is intended to span grades 4-8, some of the activities outlined here are too difficult for fourth graders. The teacher will need to select and pursue those he chooses to a depth appropriate for his class. Provision has been made in some activities for considerable depth of development. Supplementary suggestions can be found in the references given in the media listing and in the geometry strand for primary grades.

GEOMETRY

Objectives Keyed to Activities — **ACTIVITIES**

obj. 1a

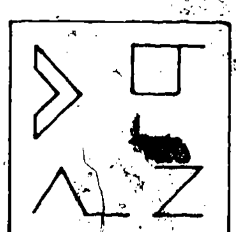
1. Draw a figure on a piece of rubber sheet such as a broken balloon and practice pulling the rubber sheet different ways to see how the appearance of the figure can be changed. Have the pupils sketch several different configurations they can produce from a single drawing as ∞ and ∞ from ∞. Some pupils might like to challenge one another by proposing tricky variations. After some individual experimentations, pupils should be asked to sketch some forms into which the original could not be deformed by pulling. Ultimately the pupils should learn that, for example, ○ can be deformed to △ and ▭ but not to ∞ or to ●, that is, ○, △ and ▭ are all equivalent. The geoboard is useful to show deformations of simple closed curves. Discussion should then bring out that simple closed surfaces such as cylinder, sphere, cube and the like are all equivalent under deformations.

obj. 1a

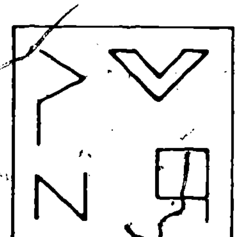
2. To show the (topological) equivalence of simple paths, direct the attention of the pupils to the way that roads are built from one side of a mountain to the other. Even though the roads may wind around the mountain in different ways, each road goes from one side to the other and does not cross itself. All such roads or paths are topologically equivalent. Have the pupils use a cone and piece of yarn or elastic thread and experiment with some of the different paths they can make.

obj. 1c

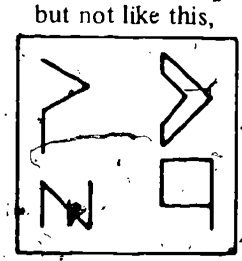
3. A transparency with a different design in each corner can be used to demonstrate reflections and rotations. The design used on the transparency should be asymmetric; if circles, stars, or such figures are used they will not exhibit all the distinctions which characterize these motions. For example, if the transparency looks like this,



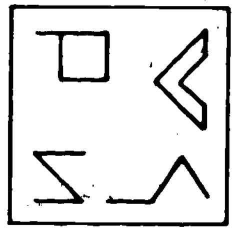
and is rotated clockwise a quarter turn, it will look like this,



but not like this,



or this.



The latter is the image under a reflection and is illustrated by turning the sheet over. The pupils will need quite a bit of experience with these motions in order to learn to distinguish them. It might be helpful if each pupil had a copy of the design so that he can handle it and imitate the motions. See activity 6.

obj. 1c

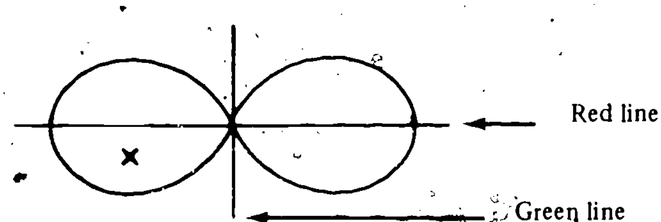
4. Potato prints can be used effectively to show translation and/or rotation. To illustrate translation, have the pupils make a potato print and then repeat the print. To illustrate rotation, use the same print (or make another) and either turn the potato or the paper each time the print is made.

obj. 1c

5. Mirror cards may be used to review or strengthen the understanding of reflection (see reference list of instructional aids). As an additional activity, the pupils may make ink blots and then use either crayons or tempera paint to add to or fill in space on each side of the blot so that one side is a reflection of the other. For further consideration, suggest to the pupils that they see if they can use a potato print to show reflection. They may be surprised to discover that they cannot. Let them use a mirror to check to see if their examples did show reflection. Additional activities may be found in books listed in the annotated references in the media section.

obj. 1c

6. To help in exploring transformations involving symmetry and rotation without introducing any theory, there are interesting games involving action which illustrate these ideas. One set of such games, called flipping games, helps to illustrate symmetrical transformations. On a sheet of acetate or cardboard, draw a figure 8 on perpendicular lines as shown in the diagram.

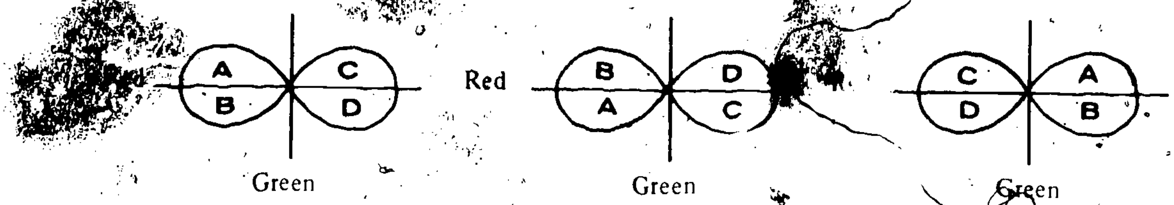


Hold it at each end of a line of symmetry and flip it over. The point X is mapped to a new position though the whole shape remains in its previous position.

One [illegible] ab[illegible]

One flip about the green line

One version of the flipper games involves 4 players. On the floor draw the figure large enough for each player to stand in one of the bounded regions which becomes his home base as illustrated.



Original placements One flip about red line One flip about green line

Mark one line of symmetry red and the other green. A fifth pupil, called the flipper, holds the acetate or cardboard on which the smaller figure has been drawn, the lines of symmetry marked red and green and the regions labeled with the players' names. If cardboard is used, it will be necessary to draw the figure, including homebase assingments on both sides, so that one is the mirror image of the other. The flipper must be sure that he holds the diagram in the same position as the diagram on the floor.

The moves of the game are as follows.

> flip – the diagram is turned over about a specified line. For instance, for a red flip the flipper would hold the sheet of acetate or cardboard at each end of the red line and flip the sheet over.
>
> rotation or turn – the diagram is turned around the center point. There are four rotations – a full-turn, a half-turn, a quarter-turn left (a left-turn) and a quarter-turn right (a right-turn).

To begin the game the flipper tells what move he will make with the shape. He may say, "I am going to make a green flip. Where will you go?" and each pupil moves to where he thinks he should be. Other problems should be posed such as, "What kind of move will take you home again?" or "I am going to do a red flip and then a green flip. Where will you be then?" More explicit directions for this activity and similar ones, as well as games for learning about rotations, are found in references listed in the media section.

When games such as the above are used for learning mathematical ideas, they should be followed by discussion to ensure that the ideas were comprehended. Some representative questions which might be asked following a flipping game are as follows.

> Is a rotation followed by a rotation the same as one rotation? (Answer yes).
>
> Is a flip followed by a flip the same as one flip? (Answer no)

The set of rotations and reflections has many of the properties of a number system– closure, associativity, identity and inverses. More mature pupils will enhance their understanding of mathematical systems by looking at this model and exploring the properties.

obj. 1b

7. Pupils seem to gain an understanding of similarity from everyday experiences. For example, model cars are similar to actual size ones, and the teacher's writing on the board to illustrate the formation of letters is reproduced similarly by pupils on their papers. Encourage the pupils to supply other examples of equivalent, enlarged or reduced copies.

obj. 1b,3c

8. a. In two similar figures, the ratio of pairs of corresponding sides or dimensions is constant. To introduce students to similar figures and related ratios, use enlargement and shrinkage of patterns. Each pupil will need graph paper on which to draw the figures as shown.

The teacher should show a figure on the chalkboard or by using the overhead projector. Ask pupils to copy the figure on graph paper using first the small squares, then the middle size squares and then the large squares.

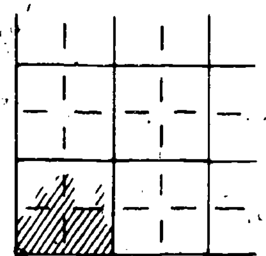


Fig. 1

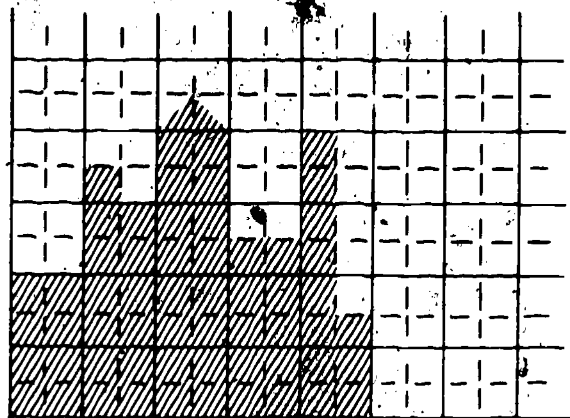


Fig. 2

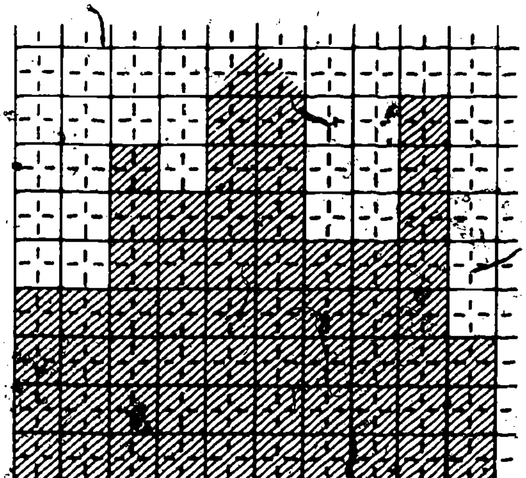


Fig. 3

Note: Smallest blocks could not be shown on these drawings.

When they finish copying the drawings ask the following questions.

Are the figures similar?

Why are they not the same size?

b. Lead the pupils to observe that each figure is ten squares wide at the bottom, four squares high on the left side and so on but that the squares are different sizes so the finished drawings are different sizes.

c. Ask the pupils if they can find a relationship between the different size squares (each medium sized square is 5 little squares on each side, and each large size square is 10 little squares on each side.) To find the constancy of the ratio, the pupils should answer the following.

(1) What is the count of small squares to the highest point of each of the three figures?

in fig. 1, there are 9
in fig. 2, there are 45
in fig. 3, there are 90

What is the ratio of the heights when comparing fig. 3 to fig. 1? Fig. 2 to fig. 1? What do you notice about these ratios?

(2) Does this ratio remain the same for other parts of the figures? Count to determine.

(3) What is the ratio of height of fig. 3 to height of fig. 2? Why should this be the case? What is the ratio of the base of fig. 3 to base of fig. 2?

d. The pupil will probably need practice in using the constancy of the ratios of similar figures. A number of problems of the following type are helpful.

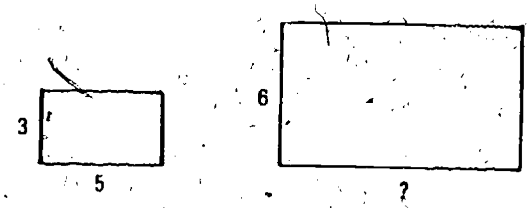


Fig. 1 Fig. 2

(1) The rectangle in figure 1 is 3 units wide and 5 units long. If the rectangle in figure 2 is similar to that in figure 1 and is 6 units wide, how long is it?

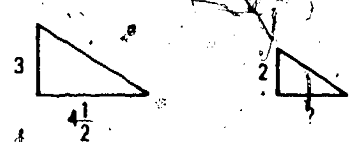
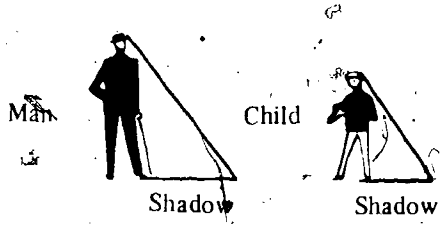


Fig. 1 Fig. 2

(2) The triangle in figure 3 is 3 units tall and 4 1/2 units long. If the triangle in figure 4 is similar to the triangle in figure 3 and is 2 units tall, how many units long is it?

e. Next the pupils might consider people and their shadows.

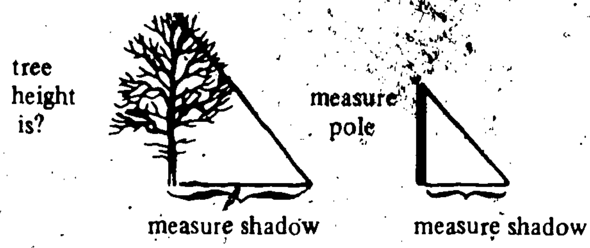


At any given time of the day, shorter people will have shorter shadows. Notice that the shadows are formed by (parallel) rays from the sun, and similar triangles ensue. Using knowledge of the ratios of similar figures pupils can solve the problem. The man in the figure is 6 ft. high and casts a shadow 4 ft. long; if the boy's shadow is 3 ft. long, how tall is the boy?

f. The pupils are probably then ready for the following activity.

Calculate the height of a tree or a building by measuring the shadow.

The children will need yardsticks and 50 foot tapes. They are to measure the shadow of the tree or building and also the height and length of the shadow of some accessible object.



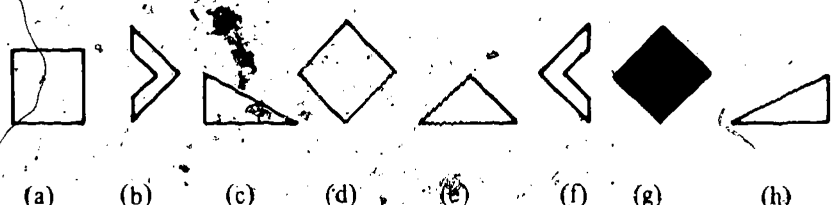
From the above measurements, calculate, using appropriate ratios, the height of the inaccessible tree.

The members of the class should work in groups with each group using a different accessible object. Have them complete this chart.

Class Groups	Object used	Length of Shadow	Height	Ratio	Length of Tree Shadow	Height of Tree
I	yard-stick					
II	Fence Post					
III	boy					
IV	girl					

Different groups may be expected to arrive at different answers for the height of the tree. Discussion of the discrepancy provides the opportunity to review the approximate nature of measurement. See the strand, Measurement.

To reinforce what has been learned about motions (rotation, translations, stretches), provide work-sheets for the pupils. Such a worksheet should show a collection of figures as illustrated.



The problem for the pupil would be to select those which are equivalent and to name the motion which makes them alike. Pupils should be expected to recognize all of the figures except (g) are topologically equivalent. Figure (d) is a rotation of (a). Figure (f) is a translation of (b) and so on.

For pupils in the upper grades, these worksheets should include angles and segments as well as simple closed curves such as squares, circles and triangles. They should become accustomed to recognizing angles of the same size in different orientations.

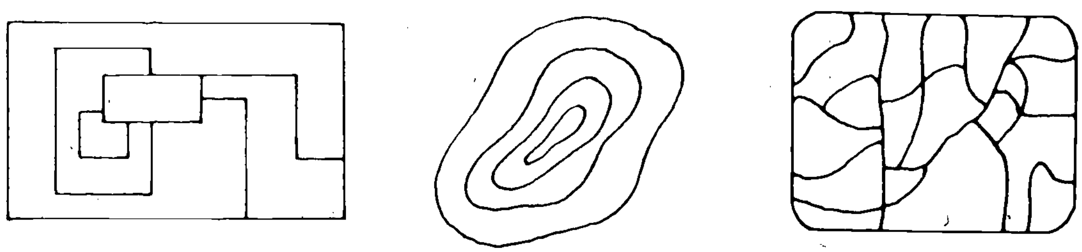
(a) (b)

For example, a rotation will make figure (a) fit exactly onto figure (b). When congruence for angles is studied later, the pupils will have had first hand experience with the idea.

obj. 1a,2b

10. An interesting topological problem concerning surfaces is that of map or crazy quilt coloring, in which neighboring countries or quilt pieces sharing a border are colored differently. One problem is to color all regions or pieces with the fewest colors possible. Generally, pupils will discover that four colors are sufficient, although it has not been proved that four are always sufficient. Suggest that the pupils attempt to design a quilt that would necessitate more than four colors.

The children may also be asked to design a quilt or map which could be made with only two colors, then three, reminding them that at least three pieces must be used and the design must cover the entire region. Some examples are given below.



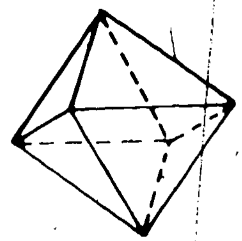
obj. 2a,1b

11. Pupils sometimes have difficulty distinguishing between surface area and volume of solids. The following activity not only offers opportunity for discovering alternative patterns for solids, it should also provide a clear distinction between points which belong and points which do not belong to these solids. Having assembled a model, the pupils should be able to visualize more easily what surface area measures.

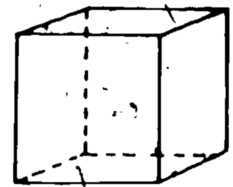
Let the pupils handle and examine models of the five regular geometric solids; they should then be asked to create their own patterns for making them. Pupils will need to try out several patterns before finding one which will work. Let the pupils make their own patterns and not use one provided by the teacher; in this way they may discover that the pattern for a solid is not unique. However, not all of the pupils should be expected to make patterns for all of the solids. Some pupils may be able to devise a pattern for only the cube and tetrahedron; others may be able to make all.

Pupils may make solids which are smaller or larger than the demonstration model, but the *similarity* of the models should be pointed out.

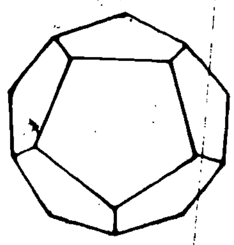
The five regular solids are as illustrated.



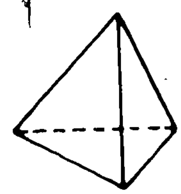
Octahedron



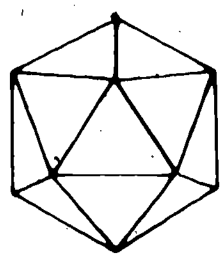
Cube



Dodecahedron



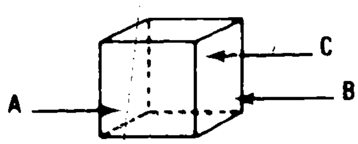
Tetrahedron



Icosahedron

obj. 2a

12. Have the pupils draw pictures of intersections of point sets. For example, have a pupil draw a picture showing all points common to faces A and B of the given cube. The response should be the drawing of the line segment common to face A and face B.



Draw the set of all points common to the top and side. Draw the set of all points common to the bottom and faces of a pyramid.

a. Draw a set of points common to a line and a triangle.
   Is there a single answer to this?
b. Draw a set of points common to a line and a circle.
   Is there a single answer to this?
c. Draw a set of points common to a plane and a sphere.
d. Draw a set of points common to a plane and a cylinder.

Other relations on point sets can be explored using questions such as the following.

Can a proper subset of a circle be a circle? (Answer no)

Can a proper subset of a circular disc be a circular disc? (Answer yes)

Can a proper subset of an angle be an angle? (Answer no)

Can the union of two rectangles be a rectangle? (Answer only if the rectangles are the same.)

obj. 2c,2d, 2f,2e

13. Classification of quadrilaterals (simple closed curves in the plane which are the union of four line segments) should be introduced by having pupils examine a collection of pictures and observe the following.

A quadrilateral may have a pair of parallel sides. (These are called trapezoid.)

A trapezoid may have two pairs of parallel sides. (These are called parallelograms.)

A parallelogram may have a pair of perpendicular sides. (These are called rectangles.)

A rectangle may have a pair of adjacent sides which are congruent. (These are squares.)

Some questions which can be raised in a natural way are as follows.

Are all parallelograms similar?

Are all rectangles similar?

Are all squares similar?

After a class has studied classification of triangles, the following questions can be raised.

Are all isosceles triangles similar?

Are all equilateral triangles similar?

Can a right triangle be equilateral?

Can a right triangle be isosceles?

obj. 2f

14. The relation *is congruent to* derives from the recognition that two point sets may be *the same size and shape*. In the activities for equivalence of point sets, the students should have observed that sets which are equivalent under rotations, translations and reflections have the same size and shape. The name *congruence* is used here, since *equality* for sets has its own meaning – two sets are equal only if they are the same set. Use the term congruent with reference to pairs of segments, pairs of angles, pairs of simple closed curves (circles, triangles, etc.), and discs. When the relation and the use of the word are both understood, the following problems may be posed.

a. Are there any congruent angles in the following figures? Identify all the pairs you can in each figure.

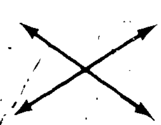


Fig. (a)

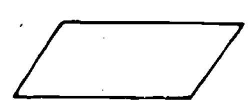


Fig. (b)

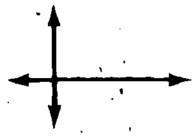


Fig. (c)

Fig. (d)

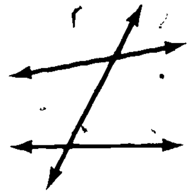


Fig. (e)

b. Can you change exactly one of the lines in figure (e) so that there will be more pairs of congruent angles? Explain.

c. Which pairs of segments in the following figures appear to be congruent?

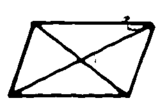


Fig. (a)

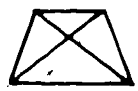


Fig. (b)

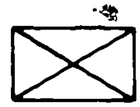


Fig. (c)

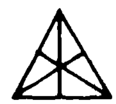


Fig. (d)

Note: There are 4 pairs of congruent segments in figure (a)
4 pairs in figure (b)
9 pairs in figure (c)
12pairs in figure (d)

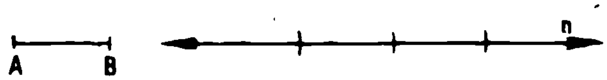
d. Which pairs of triangles in the figures for part c. appear to be congruent?

More mature students should be able to make some generalizations after answering the above questions about figures (a) through (d).

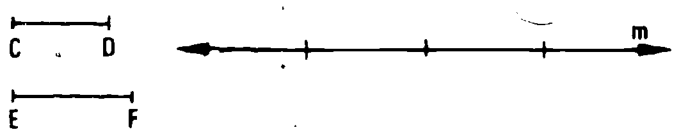
bj. 15. Classically, the geometric constructions are done with compasses and *unmarked* straightedge, the compasses (not the ruler) being used to measure length.

a. Some beginning exercises to emphasize that the ruler is for drawing straight lines and not for measuring are the following.

(1) Using your compasses, lay off on N a segment which is three times as long as $\overline{AB}$.



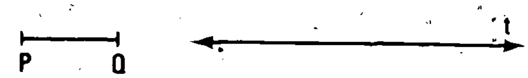
(2) Using your compasses, lay off on M a segment whose length is $\ell(\overline{CD}) + 2\ \ell(\overline{EF})$.



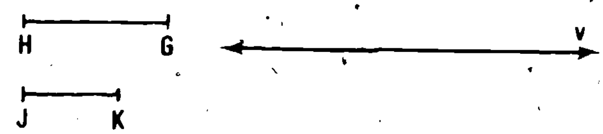
Variations of the above are easy to invent. The use of a dittoed sheet is advantageous since the emphasis can then be on the use of the compasses for measuring. Also, these should be given as real problems for the pupil – i.e., with a minimum of advance instructions. Having the faster pupils demonstrate their solutions helps communicate the expectation that these problems are solvable by the pupils.

b. When the pupils understand the appropriate use of the compasses and that construct means to use straightedge and compasses, the following problems can be proposed.

(1) Beginning on t, construct a triangle whose sides are all the same length as $\overline{PQ}$.



(2) Beginning with one side on V, construct a triangle which has one side the same length as $\overline{HG}$ and each of the other sides the same length as $\overline{JK}$.



Attempting the construction of the general triangle should lead the pupils to discover that this is sometimes impossible. The necessary experience may be teacher or pupil-motivated. However, when the impossible case arises, the pupils should be encouraged to explain why it is impossible, or alternatively, what conditions are necessary for the construction of the general triangle to be possible.

c. Using compasses and ruler, an angle can be constructed which is the reflection, translation or rotation of a given angle (called copying an angle). Once the pupil can copy a segment and copy an angle, he can construct triangles given the following information.

Three sides (with the restrictions previously discussed)

Two sides and the angle between them

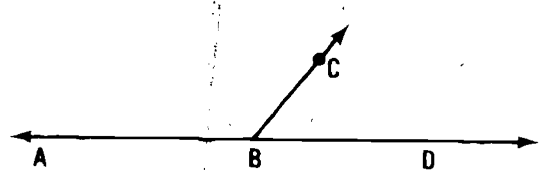
Two angles and the side between them

Have pupils compare the constructions made by all of the members of the class. What conclusions can be drawn from these comparisons?

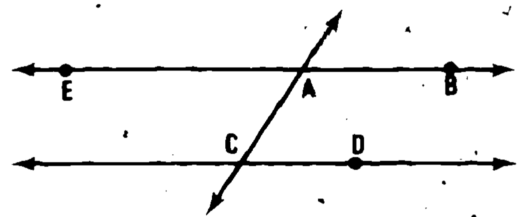
Detailed instructions for carrying out the so-called basic constructions (copying an angle, bisecting an angle, bisecting a segment, constructing perpendicular lines) are to be found in any high school geometry book.

d. According to the ability of the class, the following constructions may be done with the emphasis on the conclusions and not the accuracy of the constructions.

(1) Bisect ∠ ABC and then bisect ∠ CBD. What do you notice about the bisectors?



(2) Bisect $\angle BAC$ and $\angle ACD$. What do you notice about the bisectors?



(3) Using the same figure given above, bisect $\angle EAC$ and $\angle ACD$. What do you notice about the bisectors?

(4) In the figure below

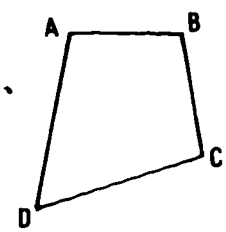
Bisect $\overline{AB}$; call the center point M;

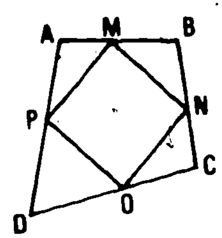
Bisect $\overline{BC}$; call the center point N;

Bisect $\overline{CD}$; call the center point O;

Bisect $\overline{AD}$; call the center point P;

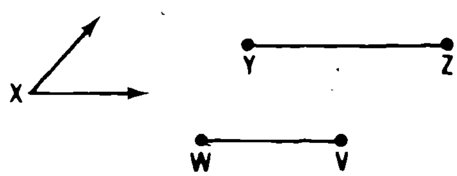
Draw $\overline{MN}$, $\overline{NO}$, $\overline{OP}$, and $\overline{PM}$. What shape do you get?





Repeat the set of constructions using a parallelogram instead of the kite.

(5) Construct a $\triangle$ to be called $\triangle ABC$ with $\angle a$ a copy of $\angle X$, $\overline{AB}$ a copy of $\overline{YZ}$, and height (to $\overline{AB}$) the same length as $\overline{WV}$. This construction will challenge the best pupils in your class.



(6) Given a circle, construct a square whose vertices are points of the circle.

(7) Given a circle, construct a regular octagon (a figure of eight equal sides) whose vertices are points of the circle

(8) Given a circle, construct a regular hexagon (a figure of six equal sides) whose vertices are points of the circle.

(9) Given a circle, construct an equilateral triangle whose vertices are points of the circle.

(10) Let $\overline{AB}$ be the diameter of a circle, and let C be any other point of the circle. Draw $\overline{CA}$ and $\overline{CB}$. What kind of an angle does $\angle\ \overline{ACB}$ appear to be?

obj. 2f

16. After a class has constructed triangles using ruler and compasses (see Activity 15), have them try to identify some conditions (other than being the image under a rotation, etc.) which will make two triangles congruent.

obj. 3b

17. Instead of using a protractor which has been purchased, pupils can make their own in order to learn how one operates. To make a protractor, have each pupil make several circles on a sheet of paper all of which have the same point as center as shown in fig. 1 below.

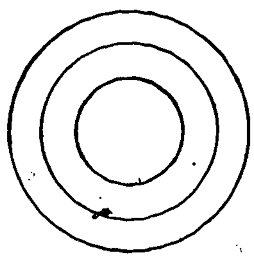


Fig. 1

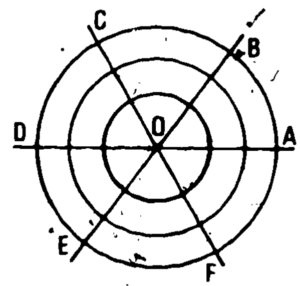


Fig. 2

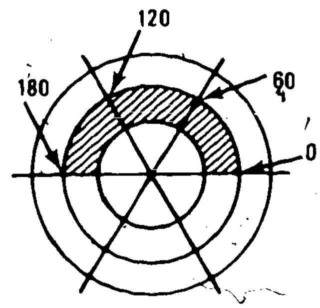


Fig. 3

Next, using compasses, partition one of the circles into six congruent parts (fig. 2), and draw a line joining each part of subdivision to the center. From previous activities, it should be apparent that, for example, $\angle$ COB is $\angle$ DOC rotated, etc., so that six congruent angles are shown. Since these are congruent, numbers should be assigned so that each of these angles can have the same number as its measure. By cutting out the interior of one of the circles, there will be a rim left (shaded in fig. 3) on which the numbers can be written. Write O where the ray $\overline{OA}$ intersects the rim, and 180 where the ray $\overline{OD}$ intersects the rim. Why 180? Because in our culture that particular assignment has been made: it is arbitrary, as are all of our units, but it is important that the pupils learn the conventional ones. With these numbers written on the protractor the point of rim corresponding to $\overline{OB}$ should be 60, the one for $\overline{OC}$ assigned 120. Notice that this will make the measures of each of the congruent angles, $\angle$ AOB, $\angle$ BOC, and $\angle$ COD the same; that is 60.

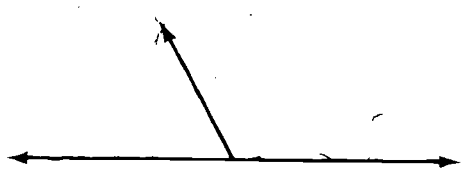
The numbering could, of course, have been started on the left instead of on the right.

By bisecting $\angle$ AOB, $\angle$ BOC, and $\angle$ COD (see Activity 15) points could be located on the rim to be labelled 30, 90 and 150. By cutting out part of the diagram the pupil has his own home-made measuring instrument for angles.

obj. 2

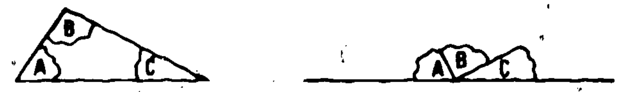
18. After both angle measure and ruler-and-compasses construction have been studied (Activities 15 and 17), some new relations can be identified and summarized.

a. The measures of the angles in a linear pair sum to 180.



105

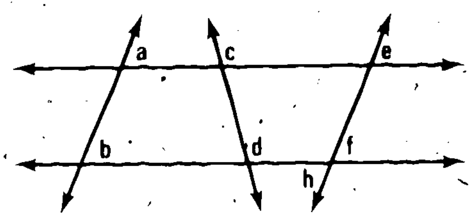
b. The sume of the measures of the angles of a triangle is 180. This can be motivated by tearing up a triangular disc cut from paper.



c. If a triangle has two congruent sides, the angles opposite these sides are congruent. Familiarity with isoseles triangles makes this an obvious observation.

d. If a triangle has two congruent angles, the sides opposite these angles are congruent. After discussion of c., a question posed to the class should bring out this conclusion.

e. Two lines in the same plane will be parallel provided that, whenever they are intersected by any third line, a pair of corresponding angles are congruent. The illustration below shows three third lines with some corresponding angles identified. (Activity 14 should make this generalization reasonable to the students).

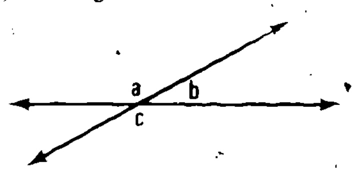


a and b are a pair of corresponding angles
c and d are a pair of corresponding angles
e and f are a pair of corresponding angles
g and h are also called corresponding angles

From these relationships, simple arguments could be given for observations already made in exploratory exercises.

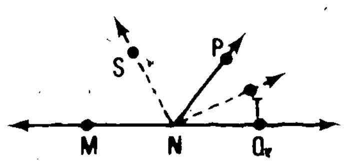
Example

f. When two lines intersect, either pair of opposite angles are congruent (figure (a) of a. in Activity 14). The argument is as follows.



Whatever size $\angle$ a is, the size of $\angle$ b must be such that the numbers sum to 180. (see a. above)
Whatever size $\angle$ c is, the size of $\angle$ b must be such that the numbers sum to 180. (again by a.)
Thus $\angle$ a and $\angle$ c must be the same size.

g. If both angles in a linear pair are bisected, the bisectors form a right angle. (See Activity 15 d.)

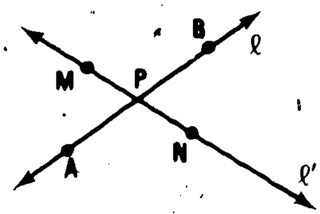


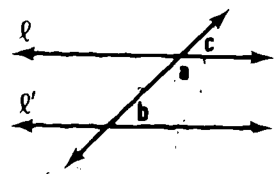
The measures of $\angle$ MNP and $\angle$ PNQ must sum to 180 (see a. above).
The measure of $\angle$ SNP is half that of $\angle$ MNP.
The measure of $\angle$ PNT is half that of $\angle$ PNQ.
Thus the measures of $\angle$ SNP and $\angle$ PNT must sum to half of 180 or 90.

h. Have the pupils bisect a pair of corresponding angles for a pair of parallel lines and note the relation between the bisectors. Then have them give an argument similar to the ones above to show that the bisectors are parallel.

For the pupil capable of moving ahead, some suggested problems are as follows.

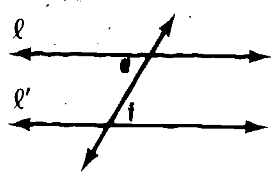
i. Let ℓ and ℓ' be any two lines intersecting at point P. On ℓ, lay off PA = PB; on ℓ' lay off PM = PN. Draw $\overline{AN}$, $\overline{NB}$, $\overline{BM}$, $\overline{MA}$. What figure results? Must this always happen? Try to give an argument to justify your answer.



j. 

What relationship appears to be true about $\angle$a and $\angle$b if ℓ and ℓ' are parallel lines?

Give an argument to justify your answer.

k. 

What relationship appears to be true about $\angle$e and $\angle$f if ℓ and ℓ' are parallel lines? Give an argument to justify your answer.

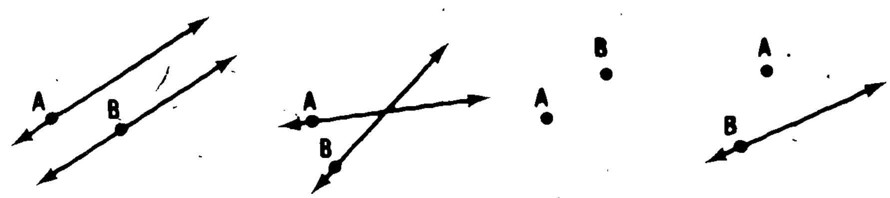
obj. 19. Have the pupils decide where all points $\frac{1}{2}$" from set A would be in each of the following cases

Case 1: A is a point

Case 2: A is a line

Case 3: A is a circle

Have the pupils decide where to find all points equally distant from A and B when A and B are placed as shown below.

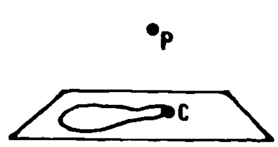


Case 1 Case 2 Case 3 Case 4

For some of these, it may be advantageous to first ask for a volunteer to find *one* point with the required property, then ask if someone could find another, etc., gradually locating *all* the points as stated in the problem. According to the capability of the class, the above problems may be solved in the plane and in space.

Some questions for more mature pupils follow.

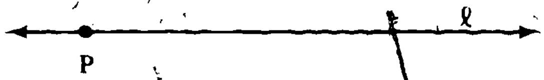
a. Describe the set of all lines through P (point P is not in the same plane as C) which contain a point of c.



b. Describe the set of all lines parallel to ℓ and containing point of c.

Case 1 Case 2

c. Describe the set of all lines (in space) perpendicular to ℓ at P.



d. Describe the set of all lines (in space) equally far from A and B.

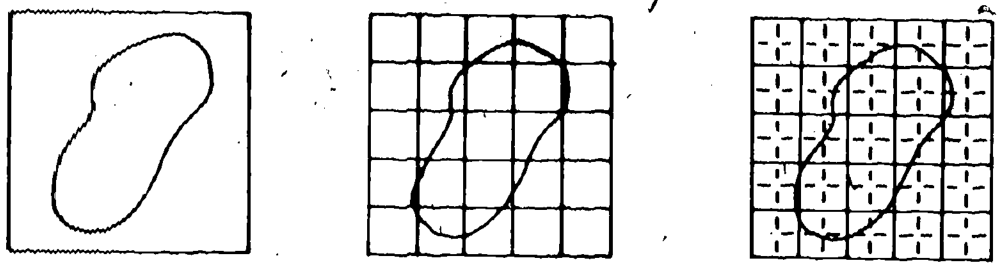
obj. 3a

20. It is important that pupils be introduced to the concept of area measurement of plane regions in a meaningful way. The area formulas, which describe computational shortcuts in special cases when there is regularity to the region (3 rows, each containing 4 units, for example, is a regular array, so that "3 X 4" computes the number of units in the array), are taken up in the strand entitled Measurement. In this strand the basic properties of measure, such as additivity, and the selection of appropriate units are developed.

Using non-standard units of measure to approximate areas of irregularly shaped discs is a good way of introducing these concepts. This gives an opportunity for the pupils to learn that the units of measurement in use today are entirely arbitrary and are a result of man's search for a reasonable way of measuring.

The following is an activity using a *non-standard* unit of measure and an irregular shaped area.

> Begin by giving each pupil or group of pupils a piece of paper on which the outline of an irregularly shaped region has been made. They are to seek means of determining how much of the paper is enclosed by the closed curve (see example). The teacher should have a number of possible units of measure ready to suggest, including a circular disc, triangular disc and a square disc, if the pupils do not suggest these. Discuss which would be best, reminding the pupils that the unit must not leave gaps or overlap, and be able to cover the entire region. They will probably decide that the square is the best. (The tiling activity in the primary grade geometry strand provides background here.) After they have selected the unit, give each pupil a sheet of transparent material on which a grid has been drawn. Since an exact measure of the shape cannot be made, there are several acceptable procedures. For example, count all squares that are entirely or partly inside the boundary; all those that are completely inside the boundary; or all those that are completely inside and those that are more than half inside the boundary. This will help the pupils in understanding that measurement is approximate, not exact. To increase precision, smaller units may be used. As in linear measurement, the unit of measure may be subdivided into smaller units, and these can be converted to the larger unit and added to the amount.



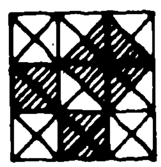
sheet given to students — using grid overlay — grid subdivided

Other irregularly shaped regions should be given the pupils for additional practice. After several of these irregularly shaped regions are used, some regularly shaped ones, as rectangular regions, might be in order, but the ideas of measurement should be emphasized.

As a follow-up, if the pupils have made models of solids (Activity 11), these can be taken apart and this measure approximated with the grid. This makes the concept of surface area more meaningful. Cardboard boxes cut so as to make one flat surface could also be used. (Introductory question might be, "Which of these two boxes do you think has more cardboard in it? How could we find out?")

obj. 3c

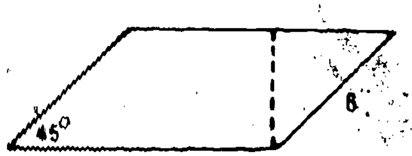
21. Pythagorean Theorem: There are many ways to motivate pupil's study of the Pythagorean Theorem, but one of the simplest is to be found on many classroom floors. If a floor tile forming the pattern ⊠ can be used, nine of the squares marked as follows would help to illustrate the relationship.

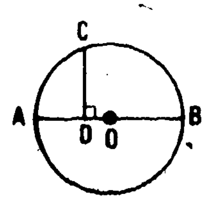


The shaded regions can be seen to be three square discs, one having a side the same length as the hypotenuse of the right triangle, and the other two having sides the same length of the legs of the right triangle. The large square disc is made up of eight tiles; each of the smaller ones is made up of four tiles. Thus the Pythagorean relationship is shown for the case in which the right triangle is isosceles. Other methods may be used for the general right triangle. Help on this may be found in books listed in the annotated references.

In application, each of the following problems makes use of more than one geometric relationship. More mature pupils may enjoy tackling them.

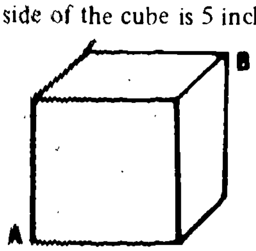
a. Find the height of the parallelogram.



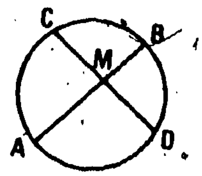


b. O is the center of the circle and $\overline{CD} \perp \overline{AB}$. If D is 1 inch from O and the circle has a radius of 6 inches, calculate the length of $\overline{CD}$.

c. If a side of the cube is 5 inches, how far is it from A to B?

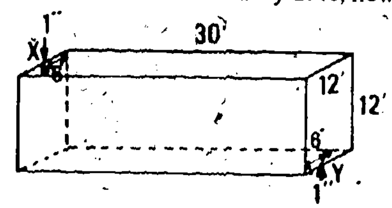


d. Complete $\ell(\overline{AM}) \times \ell(\overline{MB}) = ?$

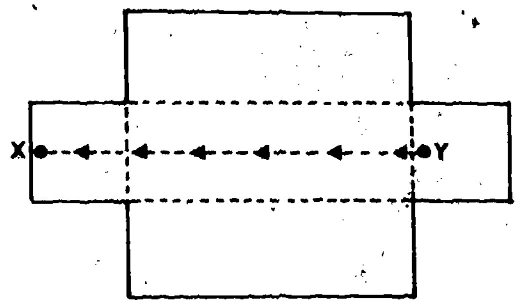


Hint– what appears to be true about $\triangle ACM$ and $\triangle MBD$?

e. A spider is at Y and wishes to catch a fly at X; how should he travel to minimize the trip?



X is midway on a wall and 1 inch from the ceiling; Y is midway on the opposite wall and 1 inch from the floor. Pupils may be surprised that the figure below shows the shortest path.

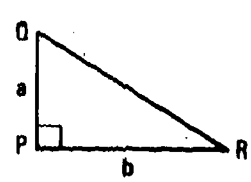


f. Find the shortest path from A to m to B in each of these cases.

obj. 3c

22. Activity 8 includes a problem involving measurement; that is, calculating the height of an object by measuring its shadow. Sometimes there is a need to determine the width rather than the height of an object when it would not be handy to use a ruler; for example, the length of a lake. The following sequence derives a way to do this.

a. Consider a triangle similar to the one shown. From Activity 8 we know this triangle has sides with lengths which could be denoted by e and f and has the property that $\frac{a}{c} = \frac{b}{f}$



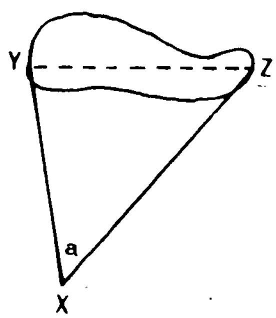
By a property of proportions the above proportion is equivalent to the proportion $\frac{a}{b} = \frac{e}{f}$. By considering other triangles similar to PQR we see that $\frac{a}{b}$ is a *constant* which names the ratio of the pair of perpendicular sides in each of these triangles.

b. Next consider a right triangle, call it ABC, which is not similar to $\triangle$PQR of the example above. Such a triangle must have an angle different in size from $\angle$ R. If the lengths of its perpendicular sides are denoted by x and y, $\frac{x}{y}$ will be a constant *different from* $\frac{a}{b}$. All triangles similar to ABC will have a pair of (perpendicular) sides, the ratio of whose lengths is $\frac{x}{y}$.

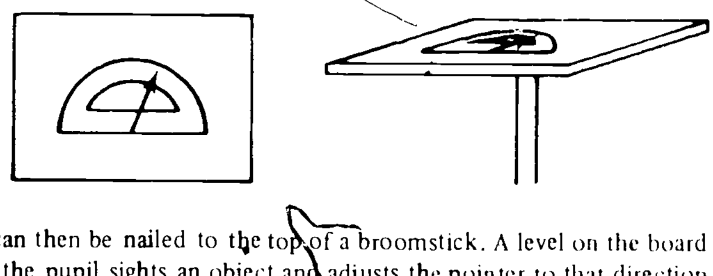
c. From the above discussion of the pairs of triangles considered, the pupils should see that all right triangles can be sorted into classes according to the size of an acute angle. In one class, for example, will be all right triangles with a 40 degree angle, in another all right triangles with a 42 degree angle, etc. These can be further characterized by the numerical value of the ratio of the perpendicular sides. These ratios have been computed and are available in tabular form called a table of *tangent ratios*. Referring to the statements in the previous exercises, $\frac{a}{b} \neq \frac{b}{a}$, so it is agreed there is an ordering; the ratio $\frac{b}{a}$, is called the tangent of $\angle$ R and the ratio $\frac{a}{b}$ is the tangent of $\angle$ Q.

d. Application – Using a surveying instrument measure $\angle$ X; suppose it is 50 degrees. A table of tangents provides the information that in a right triangle with a 50 degree angle, the tangent ratio is 1.192, thus $\frac{\ell(\overline{YZ})}{\ell(\overline{YX})} = 1.192$.

If $\overline{YX}$ is measured, then the length of $\overline{YZ}$, the length of the lake, can be calculated.



Note: Pupils can make their own surveying instrument and actually make the measurement as indicated. To make such an instrument, mount a protractor on a board with a pointer as shown.



The board can then be nailed to the top of a broomstick. A level on the board will help. To use the instrument, the pupil sights an object and adjusts the pointer to that direction. Placing pins upright on the end of the pointer and the point about which the pointer turns makes for easier sighting. The pupil should record the reading which the pointer has indicated. He then moves the pointer in line with an object on the other end of the lake and records this reading of the protractor. The difference in readings gives the size of $\angle$ X.

GEOMETRY

OBJECTIVES

The pupil should be able to do the following.

1. Select from a given set of geometric figures those which are alike
   (a) topologically (rubber sheet geometry)
   (b) under uniform stretches and uniform shrinkage
   (c) under rotations, reflections and translations
2. Identify the relations between point sets
   (a) union, intersection, inclusion
   (b) inside and outside
   (c) parallel
   (d) perpendicular
   (e) similarity
   (f) congruence
3. Apply measurements of point sets by
   (a) approximating measures of discs
   (b) measuring angles
   (c) using the ratios accompanying similar figures and the Pythagorean Theorem to solve problems
4. Solve simple problems which require the basic ruler and compasses constructions

MEASUREMENT

INTRODUCTION

Measurement is the process of relating a number to a property of an object or a set. Measure is a number which tells how many specified units have been determined.

Counting, the measure of *how many*, yields the only exact measure. In this strand, counting is used to find the number of units of measure. Activities of this type should precede those in which the pupils find measurements from reading scales of measuring devices. In using a device, the unit of measure, which has the same property of the object to be measured, is compared to the object. The resulting measure is never the same as the true measure even with the most accurate instrument.

The different kinds of measurement presented in this strand are not necessarily to be studied in the order given or separate from other strands, but rather at the same time. The development of measurement will depend upon the pupils' development in using relations, numbers, operations and properties.

Many of the difficulties in measuring that pupils have had in the past may have been due to the fact that applications were introduced too soon. The committee recommends some different approaches than have generally been used. Some of the pupils in the upper grades may need the activities recommended for the lower grades in which improvised units are used before standard units. However, the activities must be adjusted to the level of maturity of the pupils.

Time Most of the pupils will have facility in telling time by the end of the primary grades. The activities should include measuring intervals of time and telling time in which a number of units of measure are required. EXAMPLES (1) Give their ages in days, months and years. (2) Finding time intervals when given two readings in hours, minutes and seconds. (3) Telling time by various clocks, as 12-hour clock, the 24-hour clock or the nautical clock, and (4) Telling time in the various time zones, including the significance of the International Date Line.

The pupil's activities should also include finding information on the history of units for measuring time and the development of calendars.

Weight The weight of an object is the amount of force of gravity or the amount of pull of the earth on the object. Since this force is not visible, weight can be measured only by indirect means and is therefore difficult for pupils to comprehend.

The concept of weight is often confused with that of mass, which is the measure of the amount of matter. However, children are familiar with the idea of weightlessness of an astronaut, and do realize that his mass does not decrease as he becomes weightless.

Activities should be provided in which pupils actually weigh materials by using a balance. These activities should include finding weights using both the English and metric systems with special attention given to the computation involved.

The study of the historical development of measuring weight should be included in the pupils' activities. This study is not only interesting but it points out the importance of the standardization of units of measure.

Capacity and Volume These are combined as a topic since both are ways of measuring amount of space. Capacity is usually considered as the amount of space inside a container for dry materials or liquids and volume is considered as the amount of space taken up by a material or the amount of space inside a container measured in cubic units.

In the lower grades measurement of capacity and volume should be introduced prior to that of area and length, since young children's earliest experiences are with three-dimensional objects. The pupils should continue to have experiences in finding volumes and areas by counting blocks and squares in order to discover the formulas. They will soon realize that their work will be facilitated by measuring appropriate lengths and multiplying the numbers. The sequence of length, area and volume will then be used as needed in the formulas. As the pupils progress in the upper grades, they should be able to apply the formulas in appropriate situations.

Area The activities given in this section are to lead children to use standard units of square measure with meaning and to discover some of the formulas for measuring area indirectly.

Length The study of measuring length of line segments should develop similar to that of measuring volume and area. The historical development of measuring length is interesting to students and is helpful in pointing out the importance of the standardization of units of measure.

Measurement is a process or doing. It should be studied in this manner at all grade levels and should develop through a number of stages, (1) making gross comparisons of objects or sets, (2) using units of measure devised by the pupils, (3) using units of measure called standard units.

Some pupils at the upper level may need introductory activities in the first stage in order for them to realize that measurement is a comparison.

The second stage should develop from the first stage. Units of measure will be needed to relate comparisons. The first ones should be improvised or homemade ones. The pupils should make their own tables of measure so as to convert from one unit to another within the measurement of one property.

The third stage should be introduced only after the pupils have had experiences in the earlier stages and have seen a need for standard measure. Otherwise, their work will be manipulation of memorized symbols with no real comprehension of the process of measuring.

The students should make measurements themselves using various measuring devices. The devices which are first used should have few markings. As the study progresses and operations with fractions are developed, additional markings can be added or different devices can be used Activities should be included in which the children must choose the appropriate unit of measure as well as the appropriate instrument for measuring. Consideration must be given to both the property being measured and the size of unit applicable to what is being measured. Pupils at all levels should have opportunities to estimate and make approximations.

The students should have experience in measuring and performing computations in both the English and metric systems. The emphasis should be placed on working within each system and not so much on the conversion from one system to another. However, approximate relationships of the most often used units in the two systems should be discussed.

The measures of perimeters, circumferences, areas, volumes and capacities should be found by counting the units of measure. This method should be continued with activities which are planned to lead the pupils to discover the formulas. Activities of a more abstract nature can then be introduced using computation and conversion of units. Later, in the upper elementary grades precision, relative error, significant digits and scientific notation should be used in the activities.

The types of activities for measurement given in this strand are not all inclusive. Pupils at this level should study other measurements, also. Some of these are as follows.

- Speed, measured in miles per hour or feet per second
- Light brightness measured in footcandles or magnitudes of stars
- Heat the intensity using Fahrenheit and Celsius scales (formerly called centigrade scale), and the amount of heat using the units B.T.U. and calorie
- Sound the intensity and pitch
- Pressure of both air and water
- Electricity pressure measured in volts, amount of energy measured in kilowatt-hours, and rate of flow measured in amperes
- Hardness of rock or other substances measured by a scale of hardness of minerals

MEASUREMENT

Objectives Keyed to Activities

ACTIVITIES

Capacity and Volume

obj. 6,7, 8,9

1. a. Provide boxes and cubes which can completely fill the boxes. Sugar cubes and boxes about the size of those in which sugar cubes are bought would be convenient sizes of materials. Also, provide other shaped materials to put into the boxes such as marbles, small balls of cotton or crumpled paper, small triangular solids and small rectangular solids. Provide enough boxes and materials to fill them so that each group of three or four pupils will work with one set of materials.

Have each group of pupils guess the number of marbles which one ~~of~~ their boxes will hold; have them fill their box with marbles and then count the number of marbles used. Ask the pupils to repeat the procedure using the balls of cotton, the triangular solids, the rectangular solids and finally the cubes. Place the emphasis on finding how much space is inside the box; therefore, ask them to *completely* fill the box.

Ask the pupils which are the best materials to use to find the amount of space in the box. Of course, the rectangular solids and cubes would be the best since they fit together and will fill more of the space than the other materials; they are also better than the balls of cotton since the cotton can be squeezed so that the number of balls of cotton needed to fill the box is not as easy to determine as the number of blocks.

After fitting the rectangular solids or the cubes in the box, the pupils probably realized that to find the number of blocks they could count the number of blocks along the length, the number of blocks along the width, the number of layers of blocks and then find the product of these three numbers.

Ask the pupils if they can find how many blocks the box can hold without filling the box completely. They can place blocks along the length and the width, make a stack of blocks from the bottom to the top, count the blocks in each of these three sets and multiply the numbers to find the number of blocks the box would hold.

b. After experiences such as the ones described in part a., point out that the standard units of measuring volume are cubic centimeters, cubic inches, cubic feet, etc. Provide models of cubic centimeters and cubic inches, or have pupils to make some of these models. Ask the pupils to use the models and find the number of cubic centimeters and cubic inches in the boxes which were used in part a. of this activity.

Ask if they can find the number of cubic centimeters or cubic inches of volume of a box without placing any models in the box. They should probably realize that by measuring the length of the box with a metric ruler they can find the number of cubic centimeters which could be placed along the length of the box, and that the same idea would hold true for finding the number of cubes along the width and the number of layers which would fit in the box. And using a ruler marked in inches they could find the number of cubic inch models which would fit along the edges of the box. Therefore, by measuring with a ruler they could determine the three numbers to multiply to find the number of cubic units of volume of the box.

It may seem that this activity would require quite a bit of class time, but it would be time well spent. Pupils sometimes have difficulty in selecting the appropriate unit for measuring area or volume. An activity such as this one in which the measuring of volume is a physical activity would have more meaning for the pupils than activities in which volume is studied abstractly by using formulas involving measures of lengths.

c. Ask the pupils to find the number of cubic inches which a box would contain that is 1 foot long, 1 foot wide and 1 foot high. They may use boxes and models of cubes, drawings, rulers, or whatever materials they would think necessary. Also, ask them to find the number of cubic feet in 1 cubic yard and the number of cubic centimeters in a cubic meter.

obj. 6

2. Provide each group of three or four pupils with a small box which can be filled with sand and can be used as a unit of cubic measure. Also provide a box such as a chalkbox, a cylinderical container such as a coffee can and enough sand to fill these containers.

Ask the pupils how to find how many cubic units the cylindrical can will hold. Lead the pupils to realize that they can find the cubic measure by filling a container of one cubic measure with sand and pouring it into the can and repeating this procedure until the container is completely filled.

Have the pupils use the small box of one cubic unit and the sand provided, pour sand into the chalkbox until it is filled and tally the number of the cubic units used. Have them use the same cubic unit of measure to find the cubic units in the chalkbox by using the procedure of activity 1. They should find the same number of units. Then, ask them to use the cubic measuring device and sand and find the capacity of the cylindrical container.

obj. 7

3. a. Provide pupils with containers of various sizes and shapes which measure a cup, a pint, a quart or a gallon. Ask the pupils to use these measuring devices to find the number of units any given container holds. Their findings and the class discussion will enable them to see that their measurements have the same value even though they used different units. Have the children experiment and make tables of capacity. Be sure to include both a table of liquid measure and a table of dry measure. Some groups could work with liquids while others work with dry materials.

b. For activities in making changes or conversions in measurements, use recipes. This activity should lead to a discussion of the need for both liquid and dry measure.

obj. 6,7

4. Repeat activities similar to the preceding ones, using the metric system.

obj. 10

5. Have the pupils find the volume of containers with shapes of prisms and pyramids by comparing their capacity of sand or water with that of rectangular solids having dimensions comparable to those of the prisms or pyramids.

obj. 7,10

6. a. A comparison should be made for the pupils to see that one cubic centimeter is the same volume as one milliliter. This can be shown by displacement of water. Pour water into a graduated cylinder until the level of the water (the center of the curve or meniscus at the top of the water) is even with a milliliter mark. Drop an object of one cubic centimeter into the water and ask how much the water has risen in the cylinder.

b. Have the students find the volumes of cylinders and cones, following their study of area and circumference of a circle. Ask the pupils to find the number of cubic centimeters needed to fill space in a cylinder up to one centimeter and then one more centimeter and so on as done in 6a. Then some student will probably relate each of these values to the area of the circle and see that the product of this number and the number of layers, or height, will give the value of the volume of a cylinder.

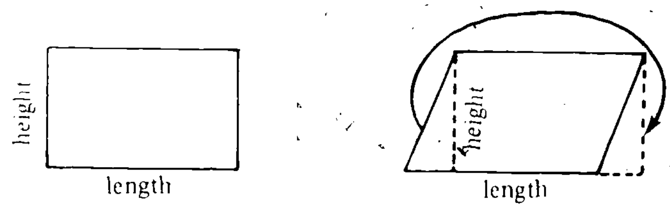
c. Have the students to use sand or water to compare the capacity of a cone and a cylinder which have the same height and same base circumference. This activity will lead to the discovery of the formula for the volume of a cone.

obj. 8

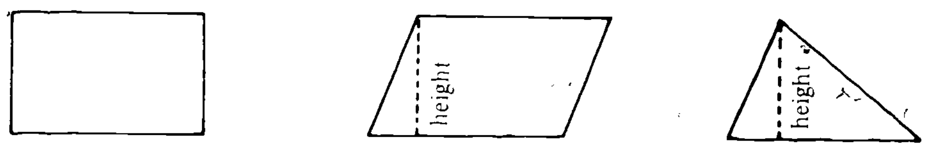
8. After pupils have discovered the formulas, they should have experiences in estimating volumes.

obj. 12 9. Have the pupils use squared paper or acetate sheets and using the squares as units find the areas of different sizes of rectangles, parallelograms, trapezoids, triangles, pentagons and other polygons as well as circles and irregular 2-dimensional shapes. The first figures used should have dimensions such that the areas of the figures will be easily determined. Ask them to find a quicker way than counting to find the areas of the rectangles and parallelograms. This activity should lead them to the formula A = bh. Some practice in finding areas by this formula sh uld follow. See the strand entitled Geometry for background information and similar activities.

obj. 13 10. Ask the pupils to use this squared paper to find the areas of regions of rectangles and parallelograms which have the same length and height. Then ask them what they believe would be a quick way to find areas of other parallelograms. This leads them to the formula for all parallelograms, A=bh.



Ask them to find the areas of a rectangle, parallelogram and triangle which have the same length or base and height. This should lead them to the formula for the area of a triangle, A=½bh.



Then have them use the squared paper to find the areas of a number of triangles which have the same base and height.

obj. 13 11. For the more advanced pupils, ask them to use the squared paper to find the area of trapezoids and to see if they can discover a way of finding the area without counting the square units of the squared paper.

obj. 13 12. a. Have the pupils use the squared paper to find total surface areas of rectangular solids, cylinders, cones, pyramids and other 3-dimensional solids. If they have studied perimeter and circumference ask the pupils to find shorter methods to determine surface areas.

b. Use activities similar to those in 4a to derive total area of cones and pyramids.

obj. 14 13. Have pupils compare area measurements using different units of measure and to set up tables of measure from these activities.

obj. 4 14. Have pupils compare area measurements of 2-dimensional figures. An example would be to compare the areas of a number of rectangles which have the same length, but in which one has twice the width of the first, one has three times the width of the first, one has four times the width of the first, etc. These comparisons should be studied from the viewpoint of the strand entitled Relations and Functions.

Length

obj. 15,16, 17

15. As the pupils gain assurance in measuring lengths using standard units, ask them to estimate lengths of various objects, record these estimations, and check the accuracy of the estimates of measuring. These measurements should include short distances such as the length of their desks and longer distances as the length of the classroom, hall or school yard. These experiences should include both English and metric units.

obj. 15,18

16. Provide drawings of polygons and circles of various sizes and ask the pupils to measure the perimeters of the polygons and circumferences of circles. Have the pupils use both the metric and English systems of measure. Lead the pupils to discover the formulas for finding the perimeter of squares, rectangles and other polygons.

obj. 17

17. Have groups of pupils read and report on the history of various units of length in both the English and metric systems. Encourage them to present their findings with demonstrations, examples of different measurements or similar visual means.

obj. 19

18. Have pupils study references on the topics precision, accuracy (relative error and absolute error), significant digits and scientific notation. Ask the pupils to report their findings with discussions or demonstrations such as stating the length of a desk in the room in inches and the distance from the earth to the moon in miles and explaining which measurement would be more precise, which would be more accurate, and why; demonstrating the effect of computations with measurements in which the use of significant digits is ignored compared with computations using the same measurements in which the rules of significant digits are used; using scientific notation to express distances in space measured in miles; and using scientific notation in computations as finding relative error.

obj. 19

19. Have pupils study and report on units of length which are related to light, such as angstrom, light year and parsec.

MEASUREMENT

OBJECTIVES

The pupils should be able to do the following:

1. Determine a time interval between two events
2. Determine final time reading given the initial reading and the time interval
3. Measure the weights of various objects using metric units and English units
4. Make reasonable estimates of weights of objects using metric units and English units
5. Construct a table of values of measuring weight using both improvised and standard units
6. Determine capacity or volume by counting improvised units, then standard units
7. Construct a table of values for measuring capacity using improvised units and using standard units
8. Make reasonable estimates of volume of rectangular solids and other three-dimensional shapes
9. Use experimentation to derive the formula for the volume of a rectangular solid
10. Derive formulas for the volume of solids other than rectangular ones by experimentation and by application of the formula discovered by the preceding objective.
11. Make reasonable estimates of the area of various regions
12. Derive the formula for the area of a rectangular region by experimentation
13. Apply the formula discovered by the preceding objective to derive formulas for the area of other regions
14. Construct a table of values of measuring area using improvised units and using standard units
15. Measure lengths using metric and using English units
16. Make reasonable estimates of length
17. Construct a table of values of measuring length using improvised units and using standard units
18. Derive formulas for finding perimeters for simple closed curves (polygons and circles)
19. Apply measurement to other fields such as science and social science

PROBABILITY AND STATISTICS

INTRODUCTION

Basic to statistics are the techniques of collecting, organizing, summarizing and analyzing data. Once the data are summarized and analyzed, the predictions that are made become the study of probability. Therefore, statistics and probability are studied together.

Because statistics is used in areas of science and social science such as insurance, marketing, astronomy and genetics, it is necessary to acquaint pupils early with the use of statistics. In addition, one of the easiest ways to develop problem solving techniques is through the collecting of statistical data that are real to the pupil. Probability and statistics are good vehicles by which the teacher causes the pupil to make choices, inferences and valid judgments.

The pupil at the upper level, as well as the primary level, should have many experiences collecting, recording and exhibiting data as these topics form the basis for the study of statistics. The pupil should then begin to have some experiences in interpreting data through the study of distributions, range, central tendency and deviation. The activities that are given in this strand are not meant to be exhaustive. Other activities may be based on such data as the top 20 records for the week, a survey of TV programs watched by most pupils, a survey of theater attendance and the number of pupils participating on various school activities.

Making predictions from the data collected necessitates the study of probability. Before formalizing the mathematical definition of probability, the pupil should engage in many game-like experiments in which he studies the chances of related outcomes. In identifying outcomes and assigning probabilities, pupils learn the concept of sample space and some counting shortcuts. These counting techniques are particularly helpful with experiments involving a great number of events.

PROBABILITY AND STATISTICS

Objectives Keyed to Activities

ACTIVITIES

obj. 1,4,5, 6,7

1. Propose a problem to the pupil in which the desired result is the data consisting of the favorite sport of all the pupils in the room. Ask the pupils to indicate the various ways that this information may be gathered. As an example, list the names of sports on the board and ask each pupil to raise his hand as his favorite sport is called, or ask each pupil to write his choice on a piece of paper and then write each choice on the board. This second method should be instigated by the teacher if the pupils do not suggest it so that the value of tallying may be discussed.

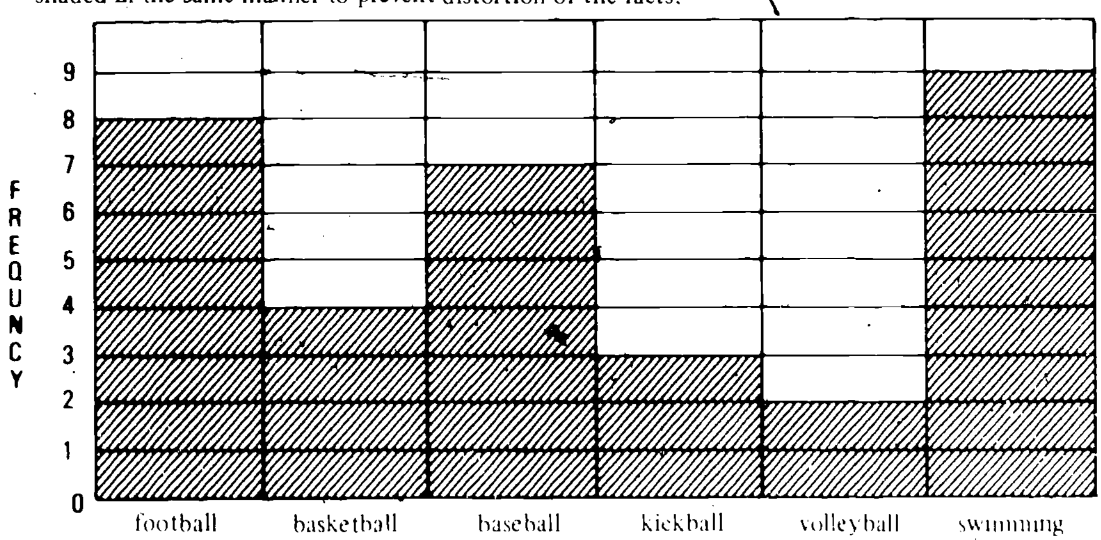
   Discuss the fact that sex distribution of the class will affect the results. For example, girls do not usually play football. Also, lead pupils to see that their ages will affect their choices.

   Have the pupils make a frequency distribution table similar to the one shown to record this data. To assure that the pupils understand the table, ask questions such as, "How many pupils liked baseball best? How many liked football best?"

Favorite Sport in Our Class

Sport	Tally	Frequency
Football	𝍸 𝍷𝍷	7
Basketball	𝍷𝍷𝍷𝍷	4
Baseball	𝍸 𝍷	6
Kickball	𝍷𝍷𝍷	3
Volleyball	𝍷𝍷	2
Swimming	𝍸 𝍷𝍷𝍷	8

Introduce the vertical bar graph (called histogram in statistics) as a means of representing data. Help the pupils to see that a graph tells the story more vividly than other methods of picturing the information. Have the pupil observe that the frequency is on the vertical scale and the categories are on the horizontal scale. Call attention to the fact that the bars should be the same width and shaded in the same manner to prevent distortion of the facts.



2. Present a set of scores from a spelling test. (Using scores from another class will prevent embarassing the pupils who made the lowest scores.) An example of a set of scores is given here for illustration.

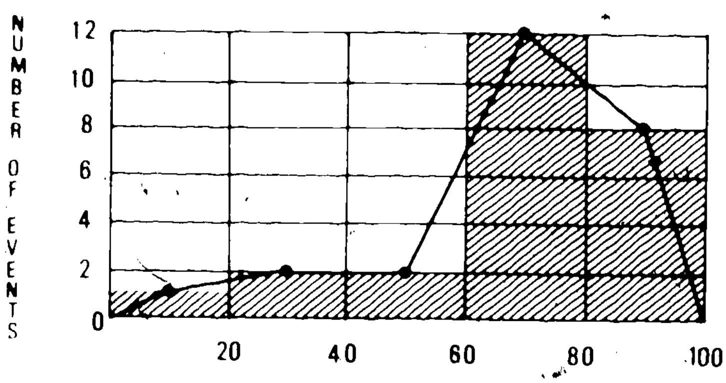
65, 81, 85, 70, 62, 18, 100, 33, 100, 40, 100, 50, 95, 60, 85, 95, 70, 80, 70, 76, 70, 75, 72, 74, 73.

Discuss with the pupils the fact that sometimes it is more feasible to use intervals in the frequency table. The scores given in the example would require a frequency table that was too long. Suggest the intervals to be used on the first graphs but lead them to make this judgment in later graphs.

Frequency Distribution Table
(Using intervals)

Interval	Tally	Frequency
0-20	|	1
21-40	||	2
21-60	||	2
61-80	𝍸 𝍸 ||	12
81-100	𝍸 |||	8

Discuss with the pupils that this information may also be pictured on histogram or frequency polygon.



To convert this histogram to a frequency polygon plot the midpoints of the intervals and connect the midpoints with 0 and 100 as shown above. The result is called a frequency polygon.

Still another way to organize scores in terms of frequencies is to consider cumulative frequencies. Have the pupil look at the original frequency chart and have him observe and record how many pupils made less than 81, less than 61, less than 41, less than 21.

Explain that this is the cumulative frequency distribution.

Cumulative Frequency Distribution Table

Interval	Tally	Frequency		Cumulative Frequency
0-20	|	1	Less than 21	1
21-40	||	2	Less than 41	3
41-60	||	2	Less than 61	5
61-80	𝍸 𝍸 ||	12	Less than 81	17
81-100	𝍸 |||	8		

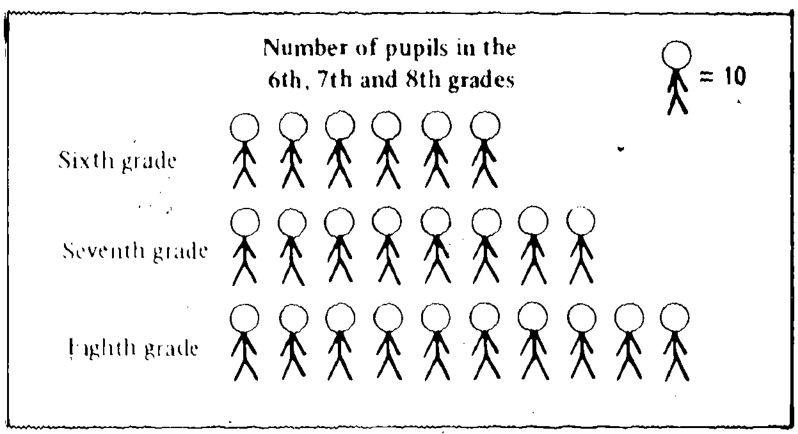
obj. 2,3, 4,5 — 3. Have a pupil ask five people at recess if they have ever had piano lessons and bring the results back to class. Discuss whether this would be an adequate sample, that is, whether this would be representative of the whole school. Ask the question, "If we ask each one in our class, would we have a sample that is representative of the entire school?" Help them to see that it probably would not be a representative sample since most children in the first, second and third grade do not take piano. They may observe that it would be representative of the upper grades.

Stress that if the pupil who gathered the information had reported that three other pupils had had guitar lessons, this information would be irrelevant information.

obj. 6 — 4. Acquaint the pupils with the fact that a pictograph is another method of picturing statistics.

Ask the pupil to prepare a pictograph using the following information. In the sixth grade there are 60 pupils, in the seventh grade there are 80 pupils and in the eighth there are 100.

PICTOGRAPH



Point out to the pupils that an appropriate scale should be chosen. This scale would be such that it will not necessitate drawing too many figures but that would permit them to draw enough to be easily read.

obj. 6,7 — 5. One activity for graphing relationships involves using a board with a small cup suspended by a spring or a rubber band. The pupil may place pennies or other objects in the cup and measure how far the cup moves down with a given number of pennies. After different numbers of pennies are used, he should then be able to predict how far the cup will drop with any given number of pennies.

obj. 6,7

6. An activity which appeals to pupils involves rolling a toy car down a ramp made of heavy cardboard. The ramp can be adjusted to various heights by using one book, 2 books and any reasonable number of books which have the same thickness. Then graph the relationship between the height of the ramp (number of books) and the distance the car travels until it stops.

obj. 6,7

7. A graph of the heights a ball bounces when dropped from various heights is a fairly simple activity for pupils. They could graph the relationship between the height of the drop and the height of the bounce.

obj. 6,7

8. Introduce the circle graph as an excellent one to use in illustrating with a picture in statistics of how each item compares with the total number. Preparing a circle graph involves many skills; therefore, it should not be used before a pupil has these skills. He could possibly interpret one or be introduced to one before he has the skills to prepare one. If he has all of the skills except those required for finding the number of degrees of each angle, he could partition the circle by folding if an appropriate example is chosen.

Stress that the first step in making any circle graph would be to find the total of the scores. If it is to show a picture comparing each part with the total, the total must be found before anything further could be done.

Using the above example the total number of pupils would be 240. Notice the total was not necessary in preparing the pictograph.

The next step would be to find what part of the total each class represents. This part may be represented as a fraction or as a per cent.

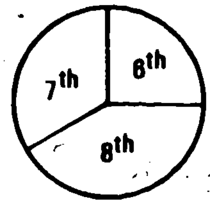
Grade	Number of Pupils	Part of the whole
6th	60	$\frac{60}{240} = \frac{1}{4}$ or $\frac{3}{12}$
7th	80	$\frac{80}{240} = \frac{1}{3} = \frac{4}{12}$
8th	100	$\frac{100}{240} = \frac{5}{12} = \frac{5}{12}$

Ask the pupil to determine the total of the parts. State this total as $\frac{12}{12}$. Point out to the pupil that since the total of the central angles is the number of degrees that should represent the total, each class would be found by multiplying each part by 360.

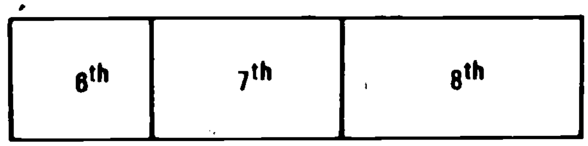
Grade	Number of Pupils	Part of the whole	Degrees to be used
6th	60	$\frac{60}{240}$	$\frac{1}{4} \times 360 = 90$
7th	80	$\frac{80}{240}$	$\frac{1}{3} \times 360 = 120$
8th	100	$\frac{100}{240}$	$\frac{5}{12} \times 360 = \frac{150}{360}$

Have the pupil observe that the number of degrees in all of the categories combined is 360.

Once the number of degrees is determined for each item, it will be necessary for the pupil to know how to use a protractor in order to place the item on the circle.



The 7th and 8th grade pupil should be introduced to the rectangular bar graph or divided bar graph. He will see that it is very closely related to the circle graph. If a bar 10 units long is used to represent the total, he would multiply each part by 10 rather than 360.



obj. 8

9. Have the pupils consider the set of scores they gathered in the previous example. Ask them to indicate what was the range of the sets of scores. Discuss with the pupils the difference that would have occurred in the range if the entire school population had been measured. The range would undoubtedly be much greater even though the measures of central tendency might have been about the same. Stress that sometimes the range may be of more interest in a set of scores than are the measures of central tendency.

obj. 9

10. As the pupils organize data in distributions, they will become aware that observed values tend to cluster. Such observation is essential for full understanding of the three recognized measures of central tendency—mean, median and mode.

Have the pupils measure the height of each person in the room and record the heights correct to the nearest inch. Then ask them to do the following.

Compute the mean of the distribution.

Find the mode of the distribution. There may be more than one mode.

Find the median of the distribution.

Discuss with the pupils that the three measures may be nearly the same or they may be quite different. Ask them which one seems to be the most representative of this set of scores. Point out that an extreme score or several extreme scores will affect the mean greatly. They will probably observe that the mean is usually the most representative except when these extreme scores exist.

obj. 9

11. For pupils to practice finding the mean, median and mode, have them work in small groups and use a set of 50 cards which are numbered 1-50. One pupil should deal 5 cards to each pupil. Each person records a mean, median and mode of his card values. The dealer then deals each person 5 cards, and each player records his scores. After 10 hands each pupil totals each of his three scores and compares his three totals with those of the other players.

obj. 9

12. Have one of the pupils obtain the facts concerning the mean income of the state or local area. Such information is probably available through the Georgia Chamber of Commerce or the Georgia Department of Industry and Trade. Discuss how the mean was computed and how extreme scores would change this measure.

obj. 10

13. Have the pupils refer to the activity in which they computed the mean of their heights. Ask each pupil how much his height deviated from the mean.

List the set of heights in order and beside each indicate the magnitude of the deviation from the mean. If a height is below the mean, place a negative sign in front of the deviation and if a height is above the mean place a positive sign in front of the deviation. This activity is not only an application of the integers, but it also provides readiness for finding the standard deviation.

obj. 11,12

14. To introduce pupils to the idea of likelihood, have them give examples of events that they are certain will happen.

Example

The sun will set in the west. A coin will fall heads or tails if one rules out the possibility of its landing on its edge.

Have them give examples of events about which there is uncertainty of occurrence.

Example

It will rain tomorrow.

Have them give examples of events that are equally likely to occur.

Example

Getting a 3 or a 5 in tossing a die are equally likely events.

obj. 12

15. Encourage the pupils to discuss the likelihood of two different events for which the chances of occurrence are not equally likely.

Example

It is more likely that an adult will eat bacon than peanuts for breakfast.

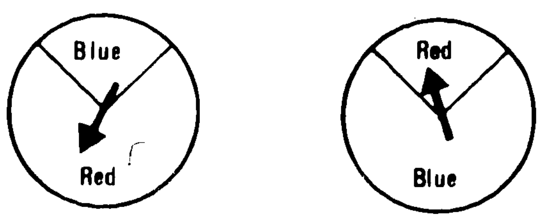
Point out to the pupil that in comparing the likelihood of these two events alternative events are excluded from consideration.

Example

Even though an adult may eat something other than bacon or peanuts for breakfast that event is excluded in this comparison.

obj. 12

16. Draw the following illustration of spinners on the chalkboard



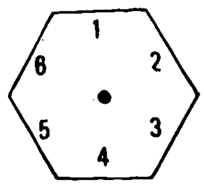
and have the pupils discuss the difference in the likelihood of getting red on first spinner and getting red on the second spinner. (Getting red on the first spinner is more likely than getting red on second spinner.)

obj. 12

17. One of the best ways to help a pupil have a feeling for equally likely events is to have him experiment with events that are *not* equally likely. There are many activities dealing with events that are not equally likely. Rolling a red cube and then a white cube, each with numerals 1 through 6 written on the sides, and recording the various sums not only gives a child a chance to practice his addition combinations, but also gives the pupil a chance to observe that getting a sum of 11 is not as likely as getting a sum of 7. He may be asked to list all the pairs of numbers and the sums he obtains in 25 times of rolling the cubes. He may then be asked to examine these pairs to see how

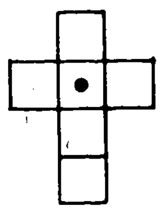
many ways he can obtain a 7. The possibilities where the first number is the red die and the second number is the white die are 1,6; 2,5; 3,4; 4,3; 5,2; 6,1. He may not have thrown each of these, but he should have obtained enough of these to aid him in listing the possibilities. When he realizes that he only has 2 ways of getting an 11 (5,6 and 6,5) he will see that the chance of getting 11 is not as great as the chance of getting 7 and thus the two events are not equally likely. He may then be asked more questions about the likelihood of various sums.

Other activities showing other variables that affect the likelihood of an event might include playing a game where the student tosses a thumbtack and records the number of times it lands with the point up, down and on its side. He may toss a paper cup to see if it lands with the open end up, the bottom end up or on its side. He may construct a simple pencil spinner by drawing a pentagon or a hexagon with sides of equal length (about 3 inches) and using a hole punched in the middle. When a pencil is inserted in the middle and the pencil is spun, the polygon will come to rest on one of the sides of the polygon. If the sides have been numbered he can record how many times it lands on each side.



If constructed correctly, this should produce events that seem to have the same chance of occuring. By making one side longer, he can design an instrument where one event is more likely to happen than the others.

Loaded cubes or similar objects are fascinating to students. A six sided cube made out of cardboard with the names of 6 different pupils may be used to play various games and this object may be loaded by taping something to one of the sides (on the inside) before the cube is put together.



A pencil may be loaded by scraping the sides of a hexagonal pencil and coloring the sides alternating colors. If one edge is scraped a little wider, the pencil will land more times on this side. In loading any device to show events that do not have the same chance of occurring, it is important not to make it so extreme that it is too obvious. One activity involving the loaded pencil would be to include a regular pencil and let the pupils determine which one is loaded after they have experimented with 25 rolls of each pencil.

A number of games for children are available commercially which require computation and game strategy. Some of these games may be used for students to find events that have an equal chance of occurring and some instances where an event has a better chance of occurring than others.

18. Pupils have a high interest in random digits and the topic is a very good one for providing the pupil with tallying experiences and counting experiences.

A spinner having 2 concentric circles with 2 digit numbers on each may be used as a source for random numbers. The pupil would be asked to spin the spinner 25 times and record the 4 digits he obtains each time. He should record the results of several spins or several pupils could record results of spins so as to compare results.

A similar experiment would be to cut pages from a telephone book and give each pupil in the class one page from the book. Each pupil would list the last 4 digits from any 25 numbers on the page.

These numbers could then be placed on the board or on strips of transparency for an overhead projector so that the various columns of 25 numbers each could be compared and the randomness of the numbers discussed.

A set of 50 cards that are numbered 1-50 can be used. Different pupils would be asked to shuffle the cards and then record the numbers in the order in which the cards are stacked. Each pupil could be given a pack of index cards so numbered and be asked to record the numbers after shuffling the cards. They could be asked to compare their lists. These cards could not only be used to discuss random digits but could be used for many other number games.

obj. 13,16

19. Involve the pupils in an activity in which the librarian for the class will be chosen by drawing a slip of paper out of a box. The name of each pupil in the class should be written on a piece of paper and placed in the box.

Suppose there are 25 pupils in the class. Ask the pupils how many possibilities there would be for picking the librarian by this method. Tell the pupils that the selection of one particular pupil is called a *possible outcome* of the experiment. Since there are 25 different selections, there are 25 possible outcomes in this experiment.

Discuss the possibility of any one pupil being chosen librarian, pointing out that this is one possible outcome out of 25 possible outcomes. Tell the pupils that this possibility could also be called the probability of his being chosen librarian. This probability is usually expressed as $\frac{1}{25}$. Have him observe that the denominator of the fraction is the total number of possible outcomes and the numerator is the number of specific outcomes, sometimes called favorable outcomes.

Introduce the idea of certainty (probability of 1) by discussing the probability of selecting someone other than himself for librarian. He will observe that P (someone else is selected) = $\frac{24}{25}$ and therefore P (he or someone else in the class is selected) = $\frac{25}{25}$ of 1. Call attention to the notation P (some event) = $\frac{a}{b}$ where $\frac{a}{b}$ represents a rational number.

Pose the question, "What is the probability that a pupil in another class, whose name is not in the box, will be chosen librarian?" This should lead to the concept that P (pupil in another class is selected) = $\frac{0}{25} = 0$. Stress that a probability measure of 0 means that the event cannot occur.

obj. 13,15, 16

20. All of the discussion concerned with choosing a librarian should develop the idea that the probability of an impossible event is 0; a certain event has a probability of 1; and all other events have a probability between 0 and 1. Prepare a box which contains three cubes– one red, one yellow, one green. Have several pupils select two cubes from the box. Any three objects of identical size and shape could be used, making certain that the pupils cannot see the one he is drawing. Be sure the pupil replaces the cube before making the second drawing. If his first drawing is a red cube and his second drawing is a yellow one, have him record the result like this– (R,Y). Other results should be recorded in a similar way. Ask pupils if there is a possibility that someone could draw a pair different from the ones recorded. Ask the pupils to record other possible results.

They are as follows.

(R,R)	(Y,Y)	(G,G)
(R,Y)	(Y,R)	(G,Y)
(R,G)	(Y,G)	(G,R)

Stress to the pupil that this set of outcomes constitutes all possible outcomes or events called the sample space of the experiment. This method of recording the events in the sample space is called listing. Have the pupil observe that since the total number of possible outcomes is 9, then

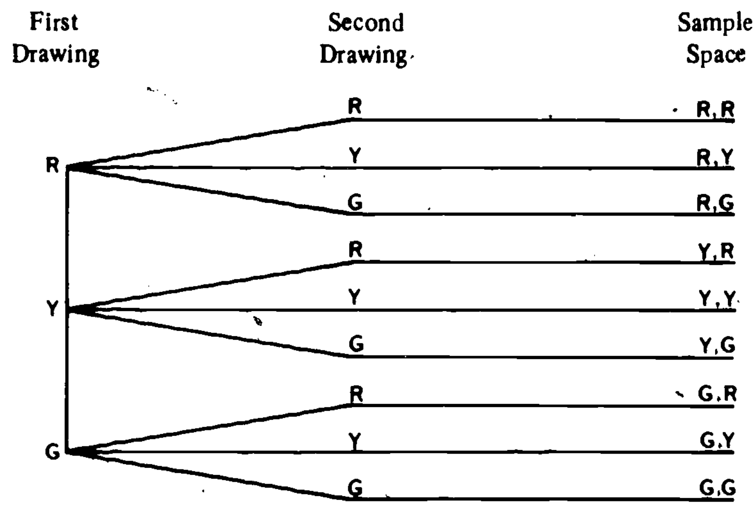
$$P(R,R) = \frac{\text{number of favorable outcomes}}{\text{number of all possible outcomes}} = \frac{1}{9}. \text{ Also, } P(Y,R) = \frac{1}{9}.$$

Call attention to the fact that in this activity (R,Y) is not the same outcome as (Y,R). Point out to the pupil that sometimes it is necessary to consider (R,Y) as different from (Y,R) and other times it is feasible that they be considered as the same outcome. For example, if Jane and Sue are picked

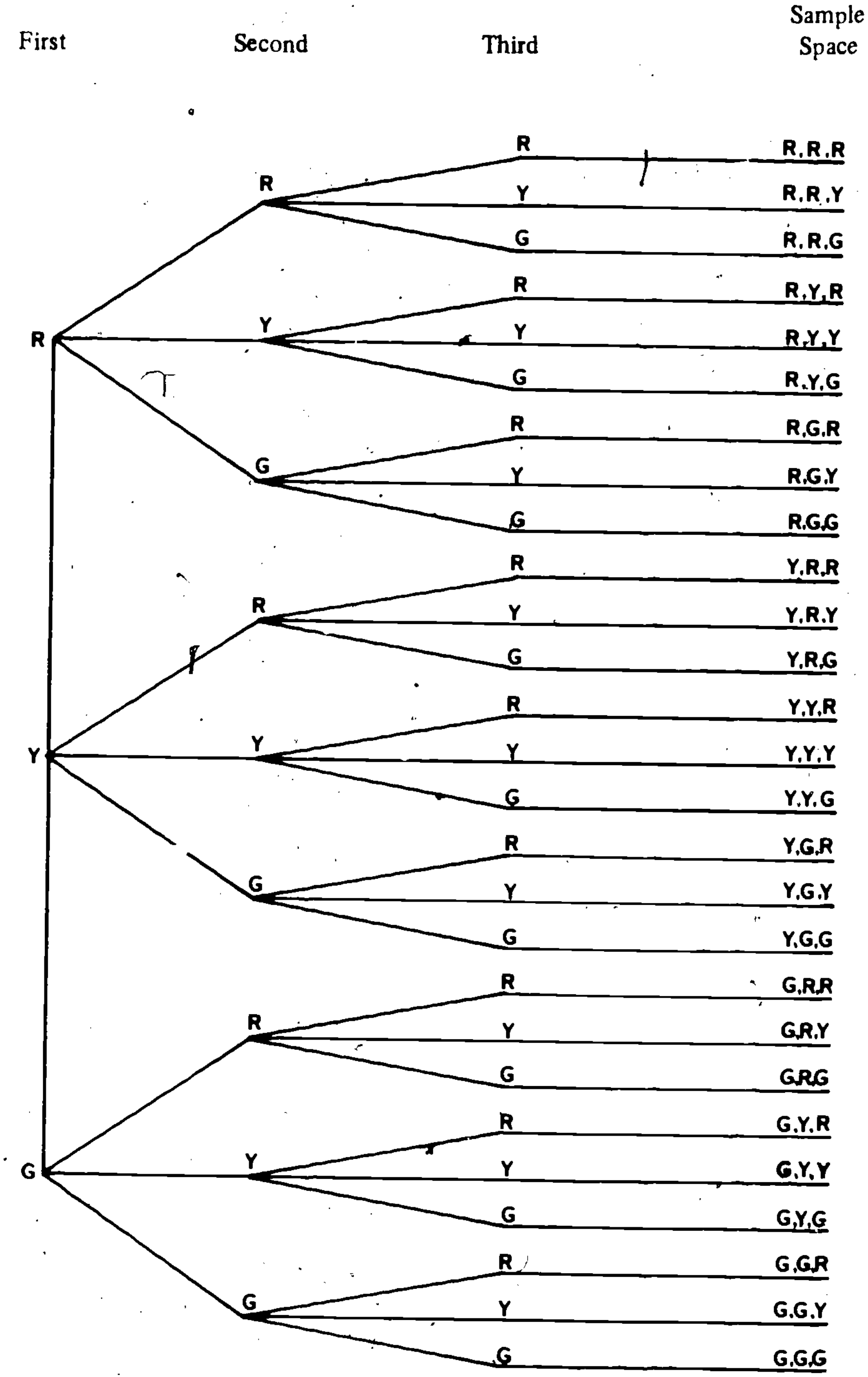
for a committee, the committee (Jane, Sue) and the committee (Sue, Jane) are considered the same. However, if Jane and Sue are to fill the places of president and vice president, then the pair (Jane, Sue) is different from the pair (Sue, Jane.)

The term *combinations* describes the first situation where order is not important. The term *permutation* applies to the second situation where order is necessary. These terms will be introduced only to the more mature pupil, although the difference should be understood by all pupils.

Tell pupils that the tree diagram is a convenient way of determining the events in a sample space. The following is a tree diagram for this experiment.



If the cubes had been drawn three times the tree would be as follows.



Lead the pupil to see that although the listing and tree diagram methods are convenient for finding all possible outcomes, they become impractical for experiments that have a large sample space.

obj. 15,16

21. Tell the pupil that there are some special counting techniques that are shorter and more efficient ways of determining the sample space of an experiment in which all outcomes are equally likely events.

One method that may be used when the sample space consists of ordered pairs (permutations) is the multiplication method or multiplication principle. In the above problem there are three possibilities for the first draw and three possibilities for the second draw; therefore, the total number of possibilities is nine. This becomes clear to the pupil if the following diagram is drawn.

First draw		Second draw		
$\boxed{3}$	X	$\boxed{3}$	=	9

Another example to illustrate this principle would be– Suppose a president, vice president and secretary are to be chosen from Tom, Ed, Bill, Paul and John. The total number of possible outcomes in the sample space will be sixty.

Pres.		Vice-Pres.		Sec.		
$\boxed{5}$	X	$\boxed{4}$	X	$\boxed{3}$	=	60

For more advanced pupils, the use of a permutation formula may be introduced. Many of the modern junior high textbooks include a discussion of this topic. It is important to note that this formula may only be used where the number of choices decreases by one as the selections are made. For example, in the example of the three cubes, the outcomes (R,R) or (Y,Y) or (G,G) are permissible. If a color is chosen on the first draw, it may be chosen again. However, in the example involving officers, once a person is selected for president, he is not eligible for the office of vice-president, etc.

obj. 15,16

22. Introduce the pupil to Pascal's triangle as one method of finding the number of outcomes in a sample space when combinations are desired rather than permutations.

Have a pupil write the events in the sample space for tossing one coin. Have three other pupils write the events in the sample space for 2, 3 or 4 coins, respectively. Have them organize the sample space into combinations. (Note that HHT, HTH, THH would all be listed as 2H, 1T). Put the results on the board in a manner similar to the following.

			1 1H, 0T		1 0H, 1T			
		1 2H, 0T		2 1H, 1T		1 0H, 2T		
	1 3H, 0T		3 2H, 1T		3 1H, 2T		1 0H, 3T	
1 4H, 0T		4 3H, 1T		6 2T, 2T		4 1H, 3T		1 0H, 4T

The pupil will observe that the total of each row is the number of outcomes (combinations) for each experiment. The recording may be extended to include another row by adding the first two members of the row above, then adding the second and third members of the row above, etc. Once he discovers this pattern, he can extend the triangle to find the number of outcomes with the tossing of any number of coins. He may then discover that the following pattern also exists.

Number of Coins	Total number of outcomes
1	2^1
2	2^2
3	2^3
4	2^4
–	–
–	–

The pupil should be informed that the sample space where combinations are necessary may be found by using a formula. Many of the modem junior high textbooks include a discussion of this topic.

obj. 15

23. Give each pupil a set of thirty-six index cards. Ask each one to write one letter of the alphabet on each of twenty-six cards (A-Z), and write one numeral on each of ten cards (0-9). Then ask each pupil to list the possibilities for tags or code names with 1 letter and 1 number.

$$\frac{\text{letter}}{26} \quad \times \quad \frac{\text{number}}{10} = 260 \text{ possibilities}$$

If this activity is used by a pupil who has difficulty with large numbers, the teacher could limit the possibilities for the letters and numbers. He might be asked to use only the vowels a, e, i, o, u and numbers 0-4. The set of possibilities would be much easier to list than the first suggestion. More advanced pupils can pursue this activity to include many ideas about the possible telephone numbers a city might have if they had one exchange, two exchanges or other numbers of exchanges.

Boys might be interested in determining the possible outcomes of a World Series. They might even be able to determine the probability of a team winning after they have lost 3 games.

obj. 17

24. To illustrate how the size of a sample affects the validity of predictions provide 5 paperbags containing marbles of 2 colors such as the following.

Bag A – 25 blue, 5 yellow

Bag B – 20 blue, 10 yellow

Bag C – 15 blue, 15 yellow

Bag D – 10 blue, 20 yellow

Bag E – 5 blue, 25 yellow

Label the bags on the underside or inside so that the pupils will not know which is A, B, C, D, E. Ask each pupil to draw 20 marbles out of each bag and replace the marble drawn each time before drawing again. Ask each pupil to tally the blue or yellow each time he draws from a bag and record the number of blues and yellows he draws from each bag. Ask each pupil to guess which bag is A, B, C, D and E by using his sample as a basis. Extend the activity by having the pupils compile the samplings and to make more valid predictions.

PROBABILITY AND STATISTICS

OBJECTIVES

The pupil should be able to do the following.

1. Identify different techniques for collecting data
2. Sort out what is relevant and what is irrelevant data
3. Demonstrate the need to collect a large enough sample of data to be representative
4. Distinguish between a biased and an unbiased sample of data
5. Give an example of how sampling affects the interpretation of the data
6. Identify various methods that may be used to record raw data and be able to select the best way to record that which he desires to interpret
7. Interpret and construct graphs
8. Determine the range in a set of numbers
9. Find the mean, median, mode of a set of numbers
10. Determine the deviation from the mean
11. Describe some events that are certain to happen and some that are certain not to happen
12. Describe some events which are equally likely to happen and other events which are not equally likely to happen
13. Assign a probability of 1 to an event which is certain to occur, 0 to an event which cannot occur and a number between 0 and 1 to any other event
14. Assign equal probabilities to equally likely outcomes
15. Count all possible ways a set of objects can be arranged under specified conditions
16. Tabulate or describe the set of all possible outcomes of an experiment
17. Make predictions based on data represented on a chart or graph

PROCESSES FOR SPECIAL CONSIDERATION

Problem Solving

Difficulties in Computation

PROBLEM SOLVING

Since problem solving is an integral part of mathematics, it is not possible to teach mathematics well and not be simultaneously solving problems. For this reason, many of the activities suggested in each of the strands in the guide are problem-oriented. The teacher, however, has the major responsibility of selecting appropriate problems for his mathematics class.

WHAT IS A PROBLEM?

In the guide a problem situation is recognized as one in which habitual response are inadequate. Word problems may in fact be only verbal exercise for some pupils. On the other hand, open sentences or exercises such as $5 + 8 = \square$ are problems for those who do not know the associated fact. To say that habitual responses are indequate is to say that recognition and recall are insufficient cognitive processes for the activity of problem solving.

The verbal exercise which begins, "John had 4 marbles and his father gave him 5 more . . .," is not a problem for the pupil who *recognizes* that a joining action is associated with addition and then *recalls* the sum of 4 and 5. In contrast, the pupil who does not know the sum of 5 and 8 is faced with a problem. In that case, the teacher's task is to ask the right questions, "What do you think it is?" "Do you want to guess?" "Why do you say that?" "How could you find out for sure?" In response to the last question, the pupil's decision will depend on his prior experiences and the performance level at which he is confident of success. He may choose 5 counters and 8 counters, recreate the joining of sets of 5 and 8 and count the members in the superset. He may place a mark at 5 on the number line and count off 8 from there; or he may reason, using well-known facts, that 8 is the same as 5 + 3. Then 5 + 8 is the same as 5 + (5 + 3). *Observing* what happens as he carries out the experiment, he *verifies* and *records* his conclusions.

Consider another example.

> Mary has 2 skirts and 3 blouse. If all of them go together, how many different outfits can she assemble?

For pupils who have studied arrays as models for multiplication of whole numbers, this example is a drill exercise in recognition and recall. For pupils who have studied only repeated addition as the model for multiplication, it is a problem.

The selection of problems for mathematics classes becomes, then, a matter of selecting situations in which new relationships must be found among well-known facts or in which the context is so ~~novel~~ that pupils must explore, observe, make conjectures and test their conjectures.

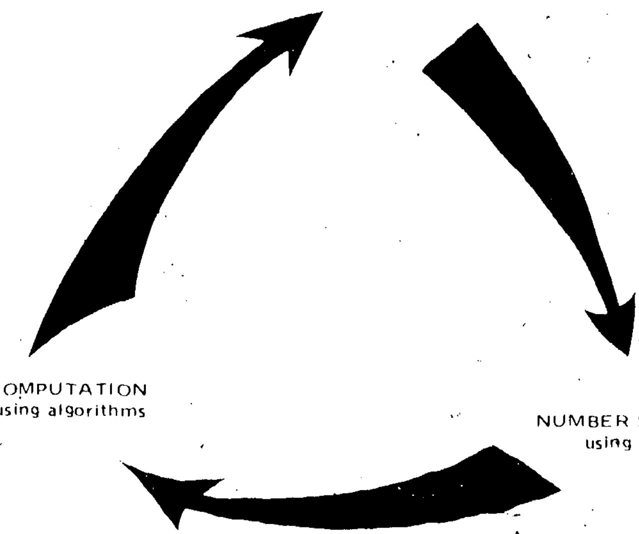
A Classroom Environment for Problem Solving

Given a problem situation, the role of the teacher is to provide an encouraging climate in which pupils feel free to make guesses, checking out their guesses at whatever performance level they are successful, talking about their observations and making decisions which are defensible in terms of the conditions and data given. Teachers can aid pupils in developing the right sort of guessing by asking questions such as the following. "What do you think? Which of these two cans holds more water?" "What do you think? Which is the greater number, $\frac{7}{8}$ or $\frac{7}{9}$?" Following the recognition or recording of pupils' conjectures, the teacher should ask, "How can we find out?" After pupils have carried out the experiment and observed what happens, the teacher should ask perhaps the most important question of all – "Why did we think the answer was one thing when in fact it was something else?"

Collecting opinions from pupils at the outset may seem unusual in a mathematics class. However, it can give the teacher valuable insight into their thinking as well as focus attention on the nature of the problem at hand. The selection of problems, the questions asked by the teacher and reactions to pupils' guesses are all very important in developing and improving competencies in solving problems. Since some pupils are discouraged by being told that they are wrong, an important contribution from the teacher is his expectation that they *can* solve problems. His confidence can encourage easily frustrated pupils to try again and can be contagious. Persistence and confidence are characteristics of good problem solvers.

Other Factors Related to Success in Problem Solving

In addition to teaching guessing and encouraging flexible approaches to problem solving, the teacher needs also to provide for his pupils' acquisition of knowledge of fundamental concepts in mathematics, skills in computation and reading and other interpretive abilities.



Knowledge of fundamental concepts is acquired through a variety of planned experiences. Consider the word problem.

> Three boys were playing marbles and two other boys came to play with them. How many boys are now playing marbles?

When difficulties occur with exercises of this type, the difficulty is probably not one of reading nor one of computation, but one of not knowing the meaning of the operation called addition. Help for this difficulty involves the right kind of initial presentations of the operations concepts. It is for this reason that a distinction has been made in the guide between operation and computation. Teaching addition is different from teaching how to compute the sum; the former encompasses the *when* and *why* one adds, whereas the latter does not. If sufficient attention is given to the meaning and effects of operations, pupils should have less difficulty in recognizing verbal descriptions of situations for which the different operations are appropriate. On the other hand, if computation is given the major emphasis, it should not be surprising that pupils fail to recognize a situation which calls for a particular process. Teachers should see the strand on Operations, Their Properties and Number Theory.

Teachers also need to be alert to possible difficulties associated with the no-action situations. The joining of two sets of boys described in "Three boys were playing marbles and two other boys came to play. . ." is identified with the operation of addition. In the verbal description, the action of joining two sets is explicit. However, some pupils may have difficulty with verbal problems of the following type.

> Susan and Jane live on adjacent blocks. Susan counted 7 brick houses on the block where she lives, and Jane counted 5 brick houses on her block. How many brick houses are there in the two blocks?

The two sets of houses cannot be joined physically Nevertheless, the reader must think about the two sets as *joined* in some superset and recognize that the situation described is an additive one.

Corresponding to the other operations are the same categories of *action* and *no action* problems. Quite often, it is helpful for pupils to recreate problem situations with manipulative or pictorial aids. Some will need physical experiences with counters or toys, while others will need only pictures or diagrams which they can draw. Problem difficulty has been assessed along the aids/action dimension as follows.

	Problem Situation	
Aids	Action	No Action
Physical	Easiest	
Pictorial		
None		Hardest

In light of the above discussion, the usefulness of cue words in verbal problems becomes highly questionable. The better strategy seems to be one of developing the concepts of operations and the concurrent development of language – in both oral and written descriptions – associated with the concepts. In the guide, it is suggested that pupils be given an equation such as $3 + 4 = \square$, or $3 + \square = 7$ and be asked to make up a word story with which the equation could be associated. By encouraging a diversity of responses, a teacher can aid pupils in learning many different settings and situations for which a particular operation is appropriate. At the same time, he is helping them develop the language skills one needs for reading and talking about mathematical ideas.

Consider the following word problems.

(1) John has 6 pennies in his pocket *and* 3 pennies in his lunchbox. If he *puts* them together, *how many* pennies does he have?
(2) John has 6 toy cowboys *and* 3 toy horses. If he *puts* one cowboy on each horse, *how many* cowboys will not have a horse to ride?
(3) Before school started in September, Mary's mother made her 6 new blouses *and* 3 new skirts. If she could *put* the blouses and skirts together in any combination, *how many* new school outfits would Mary have?
(4) Mary has 6 roses *and* 3 vases. *How many* roses could she *put* in each vase so that there are the same number of roses in each vase?

In each case, the word "and" serves its logical role as a conjunction and does not indicate the action involving the given sets. In each case, the word which suggests the action is "put(s)." In the first problem, two sets are put together to create a superset. In the second problem, two sets are put together for the purpose of comparing them. The putting together involves the action of matching objects from one set with objects of another set in a one-to-one correspondence until all the objects in one set are exhausted. In the third problem, the putting together involves the pairing of each object from one set with each object of the other set, that is, the creation of a third set in which the elements are outfits, or pairs (one skirt, one blouse). In the fourth problem, the putting together of the two sets involves the action of establishing a many-to-one correspondence of the given sets. In each case, the question is "How many . . .?" There are no single, clear-cut word cues for corresponding mathematical operations.

Neither the ability to recall the number facts nor skills in reading as measured by standardized tests are sufficient cognitive factors for solving these problems. One must also possess an ability to recognize that certain kinds of action (real or implied) are associated with particular mathematicial operations. The successful problem solvers, in this case, are those who associate addition with the action described in problem 1, subtraction with the action described in problem 2, multiplication with the action described in problem 3 and division with the action described in problem 4.

Teachers should provide a variety of experiences in which pupils can talk about their understanding of the operations. For instance, pupils may be given the four verbal problems above (or some similar set) and either of the following sets of corresponding expressions.

(a)
(1) the sum of 6 and 3
(2) the difference of 6 and 3
(3) the product of 6 and 3
(4) the quotient of 6 and 3

(b)
(1) $6 + 3$
(2) $6 - 3$
(3) 6×3
(4) $6 \div 3$

They should be asked to discuss (and demonstrate, if necessary) *why* the corresponding expressions give the correct solutions.

Other techniques teachers may use to help pupils improve their problem solving abilities through increased understanding of the fundamental operations are these.

- Use problems without numbers.
- Have pupils act out problem situations and solutions.
- Have pupils write only the corresponding number sentences for verbal problems (but not the answers).
- Have pupils make diagrams or drawings to illustrate their solutions.
- Use problem with insufficient data or irrelevant data.
- Have pupils make up problems for given pairs of numbers (e.g., formulate four different problems in which the numbers 24 and 8 are used, with each problem solution to involve a different operation on 24 and 8).

Skill in computation is, of course, a necessary condition for getting correct answers, and teachers should have pupils test the reasonableness of their answer as a check on their computations. Such tests may also provide checks on the appropriateness of processes selected. (For further discussion on skills in computation, teachers should see the section on Difficulties in Computation.) The tediousness of computation should not be allowed to obscure the principle inherent in a particular verbal exercise. For instance, if pupils are studying the principle related to finding the arithmetic mean of a sample of n measurements, the emphasis of classroom activities should be on the mathematical principle. In this case, once the pupils have identified the procedure to be used, it is helpful to use an adding machine to take care of the tedious task of summing all the measurements.

Reading and other interpretive skills should be developed as an integral part of the mathematics program, with special attention given these skills in class work with verbal problems. In addition to the techniques previously suggested for developing language and reading skills, the following suggestions should also prove beneficial.

Help pupils learn to use their textbooks. Take time to read with them certain narrative sections of the book, including the preface, if there is one. Take time to note the table of contents and to teach the use of the index. If the publisher has a glossary of mathematical symbols and word definitions, make it accessible to pupils. Encourage and help pupils find out who the authors are and secure biographical information about them.

Have pupils read library books and reference materials related to mathematical ideas. For instance, in the upper grades, when studying measurement, have a committee of pupils research a topic such as "The History of Linear Measures" and report their findings to the class. Identify science-related materials and problems (which involve mathematical principles and skills) for pupils to read and solve. Identify charts and graphs from the social studies program which require interpretive skills learned in mathematics. Help pupils to know that mathematics is a lively, purposeful and dynamic force in their culture.

Provide specific instruction in quantitive vocabulary. The study should include the role of prefixes and suffixes with word roots which occur in mathematics, as in the words polygon, polynomial, triangle, trinomial, equalities, equidistant. The study of word endings should be included, as in the words polygonal, rectangular, radii. The study of prefixes, suffixes, roots and endings which have special meanings will facilitate the learning and spelling of new terms.

Have pupils formulate and write verbal problems, or have them jointly discuss how to explain a selected mathematical principle and write the verbal explanation on the board. Class discussions of the written materials should lead to refinement and classification of the language used. Encourage pupils to avoid redundancies and ambiguious descriptions. An occasional mathematics period devoted to developing skills of conciseness and precision in language is well-spent.

Have pupils read selected word problems for the purpose of finding specified information. What is the question? Describe the setting. Who is involved? What sets of physical things are involved? What is the order of events which occur over a period of time? What is the relationship or action described? Errors or inabilities in comprehending what they read may be due to the pupils' failure to read for the express purpose of noting details. Explanatory or descriptive materials which deal with mathematical ideas are marked by a conciseness which is not characteristic of reading materials in general. Know the reading abilities of pupils and select or adapt materials to their ability levels.

Present problems orally. Provide for the poor reader by using a tape recorder to record word problems. There are many opportunities for engaging all pupils in problem-solving activities through open discussion, through manipulation of physical models and through the use of pictures and diagrams. No pupil need be denied learning experiences in problem solving.

In addition to selecting and adapting instructional materials and procedures to the abilities of his pupils, the teacher himself can serve as a model in the use of language. For instance, the simple written expressions, "the product of 8 and 9" or "the sum of 8 and 9" are difficult for those pupils who have always read "8 X 9" as "8 times 9" or "8 + 9" as "8 plus 9." Teachers can easily provide experiences with such verbal language expressions through their own use of them in writing at the board and in reading symbolic expressions such as "8 X 9 = 72" as "The product of 8 and 9 is 72." Instead of asking pupils to "find the answer to...," the teacher could ask them to "find the sum of..." or "find the quotient of..." Consider the following ways of writing one simple fact in mathematics.

(1) $3 + 4 = 7$

(2) $7 = 3 + 4$

(3) $\begin{array}{r} 3 \\ +\,4 \\ \hline 7 \end{array}$

(4) Three plus 4 is equal to 7.

(5) Seven is the same as three plus four.

(6) The sum of 3 and 4 is 7.

(7) Seven is the sum of 3 and 4.

(8) Four more than 3 is 7.

(9) Seven is equal to 4 more than 3.

(10) Four added to 3 is 7.

(11) Seven is 4 added to 3.

If children hear and use the verbal variations in (6) through (11), they are more likely to be able to read such statements with understanding.

The ability to read any verbal problem with understanding and insight and find the solution is a complex ability which encompasses several cognitive factors and processes. There is no one step-by-step procedure which is best for every problem-solver. However, the following procedure is suggested as a guide for general classroom use.

- Tell a short story. That is, have the pupil read the verbal problem and revise the narrative so that it is in the forn of a short story.
- Select the mathematical operation associated with the problem structure.
- Write the number story or open number sentence corresponding to the story.
- Solve the numerical problem.
- Answer the question asked in the verbal problem.
- Check to see if the solution is reasonable.

DIFFICULTIES IN COMPUTATION

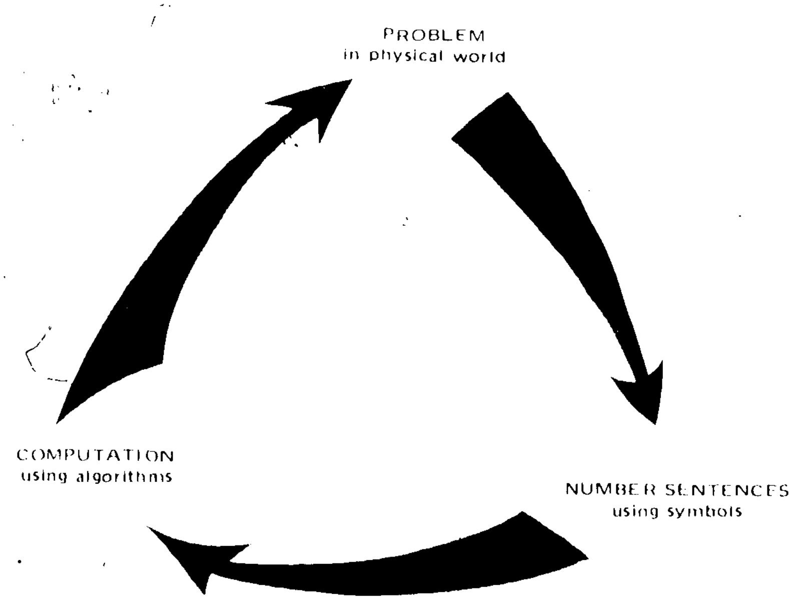
INTRODUCTION

In this section, the place of computation in the mathematics program is discussed. The committee has examined specific types of computations which students find difficult and has made some suggestions.

In the strand Operations, Their Properties and Number Theory a distinction is made between operations and computation. Computation is the processing of numerals. The process yields the name of that number which is assigned to a pair of numbers by the operation. For example, the operation of addition assigns a single number called the sum to the pair of numbers 17 and 18. By the process of computation, one determines that the name of this sum should be 35 in the decimal system of numeration.

In order for children to obtain skills in computing, they must first have an understanding of the operations. As described in the strand Operations, Their Properties and Number Theory, each operation should first be related to a physical situation, then illustrated through pictorial representations and number line activities and recorded with numerals and operational symbols.

The abstraction of the operations should be pulled from the physical world so that later in problem solving there will be no trouble when one also moves from the abstraction back to the physical world. The operations need to be clearly understood in order to organize and set up problems in symbols. The place of computation in solving problems may be seen in the following graphical illustration.



In the lower grades, teachers typically make a practice of offering children experiences with physical models associated with the operations and having them record the results using mathematical symbols. However, in order to develop any of the computational algorithms, the pupil needs to use physical models for the operations that emphasize bundling collections into tens, hundreds, etc. For instance, the pupil sees that a collection of 9 objects joined to a collection of 6 objects yields a collection of (6 + 9) objects. He should then actively experience the bundling of this collection according to the ten's rule, or decimal coding scheme. The act of bundling and then recording the sum as one-five (1 bundle and 5 singles) should precede naming the sum "fifteen." Early experiences with small sums such as these are necessary for children to develop insight into computation as the processing of decimal numerals. It should be recognized that this bundling is the physical analogue of those maneuvers called *renaming* which occur again and again in the computational algorithms.

With larger numbers, one or more of the addends may need to be represented with bundles. For example, in determining the sum of 17 and 6, the 17 should be represented by 1 bundle of 10 and 7 singles. When the collection of 6 is joined to this, the resulting collection cannot be assigned a decimal name until more bundling is done. In the example in the second paragraph of this section, the sum 17 and 18 should be represented by 1 bundle of 10 and 7 singles joined with 1 bundle of 10 and 8 singles. Again the decimal name of the sum cannot be assigned until all possible bundling has been completed. The sum should then be recorded as 35. The pupils should work individually on many similar examples with physical objects before making the transition to using pencil and paper exclusively.

The stage in which children move from the basic concepts of the operations to the final computational skills is called "why (understanding)." And those skills which the children finally acquire to find the result with maximum speed are termed "how (skills)." The following examples show these stages.

	Why (Understanding)	How (Skill)
Problem 87 − 48 = ☐ Understanding the operation subtraction, decimal notation and regrouping, and knowing the number facts	$$\begin{array}{rcl} 87 = 80 + 7 & = & 70 + 17 \\ \underline{48 = 40 + 8} & = & \underline{40 + 8} \\ & & 30 + 9 = 39 \end{array}$$	$$\begin{array}{r} 87 \\ \underline{-48} \\ 39 \end{array}$$
Problem 91 ÷ 7 = ☐ Understanding the meaning of division and the operation of substraction, and knowing the number facts	$$\begin{array}{rl} & \text{counting} \\ 91 & \\ \underline{-7} & 1 \\ 77 & \\ \underline{-7} & 2 \\ 70 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 14 & \\ \underline{-7} & 12 \\ 7 & \\ \underline{-7} & 13 \end{array}$$ so 91 ÷ 7 = 13 Later $$\begin{array}{r} 3 \\ 10 \\ 7\,\overline{)\,91} \\ \underline{70} \\ 21 \\ \underline{21} \\ 0 \end{array}$$ so 91 ÷ 7 = 13	$$\begin{array}{r} 13 \\ 7\,\overline{)\,91} \end{array}$$

The above activities allow children to discover their own patterns for computation. Since there is no single algorithm for any of the operations, the teacher should be willing to let children use any mathematically sound process they wish.

The development of computational skills continues to be one of the important goals of mathematics education. When pupils have difficulty in computation, the teacher will find it beneficial to diagnose the difficulties. Although diagnosis is time consuming, results will be worth the expenditure of time. Early diagnosis will prevent pupil frustration, failure and consequent lact of effort. If pupils do not understand the algorithms, additional problems of the same type will not prove beneficial. However, if they can see why they made mistakes and can learn the correct procedures, additional practice will then be helpful. Mistakes may result from lack of understanding basic facts rather than from failure to understand algorithm. Acquiring skill in basic facts requires practice, repetition and drill, but the teacher should use a wide variety of drill techniques.

An all-inclusive treatment of computational difficulties is beyond the scope of this guide. The examples given, however, will serve as models for diagnosing difficulties which the teacher will encounter in his own classroom.

Typical Computation Errors	Suggestions

I. Subtraction of whole numbers

A. Jim wants to buy a popsicle that costs 10¢. He has 7¢ in his pocket. How much money must he take from his piggy bank to be able to buy the popsicle?

Incorrect	Correct
17¢	3¢

The pupil does not realize that there are uses of subtraction other than take away. The example shown at the left is one example of how many more are needed.

Exemplify different models for subtraction. Some of these a (1) Take away. Use a set of 9 objects. Remove 3 objects. How many are left? (2) How many more Use a set of three objects. Ask, "How many more are needed to make 9?" (3) Comparison -- Compare a set of 9 objects with a set of 3 objects. Ask, "How many more a n th largest set?" (4) Inverse of Addition -- Write + ☐ = 9 or ☐ + 3 = 9. Ask, "What number would make this a true sentence?" See the strand entitled Operations, Their Properties and Number Theory.

Many errors in subtraction which pupils make throughout the elementary grades could be avoided with the use of proper vocabulary. On beginning the use of the symbol, "–," in subtraction, translate it "minus," not "take away." For example, do not read 9 – 3 as "nine take away three," but read it as "nine minus three." Also do not over-simplify in explaining by saying, "You always take the smaller number from the larger," since this leads to errors in computation later as illustrated in the next example.

B.

Incorrect	Correct
$\begin{array}{r} 25 \\ -8 \\ \hline 23 \end{array}$	$\begin{array}{r} 25 \\ -8 \\ \hline 17 \end{array}$
$\begin{array}{r} 304 \\ -82 \\ \hline 322 \end{array}$	$\begin{array}{r} 304 \\ -82 \\ \hline 222 \end{array}$

The pupil may understand the subtraction operation but not comprehend the subtraction algorithm.

This difficulty is due to a lack of understanding of place value or of different ways of naming a number. A pupil having such difficulty should have experience using place value devices with counters. These problems would require the use of counters bundled into tens and hundreds. From the use of manipulative

materials the child should realize that in subtracting 8 from 25, he may represent 25 as 10 + 10 + 5 or 10 + 15. The teacher should lead the pupil to see that the logical choice here would be 10 + 15 for two reasons. He needs 8 or more in the ones column, and since the second column contains only multiples of 10, he must use one 10 from that column to combine with the 4 ones. Pupils should be encouraged to use methods other than the standard algorithm if they are better understood. Some alternate methods are as follows.

(1) Number line

Marking the 8 as an addend and 25 as the sum, and finding the missing addend.

(2) Counting on or counting back (similar to the complement method)

25 − 8 = 15 + (10 − 8)
which is 15 + 2
which is 17
so 25 − 8 = 17.

or

25 − 8 = 5 + 20 − 8
which is 5 + 12
which is 17
as 25 − 8 = 17

(3) Algorithm of integers (where appropriate)

$$\begin{array}{rl} 24 & \\ \underline{-8} & \\ -4 & \text{(by } 4 - 8 = -4) \\ \underline{20} & \text{(by 2 tens} - 0 \text{ tens} = 2 \text{ tens or } 20) \\ 16 & \text{(by } -4 + 20 = 16.) \end{array}$$

(4) Other methods which pupils may devise. They should be encouraged to analyze their methods to make sure that they are mathematically sound.

II. Division of Whole Numbers

For background for the division algorithm each pupil should progress through the following in sequence. Some pupils may move through this sequence more rapidly than others depending upon their understanding of each stage. If a student is having difficulty with the final algorithm, he needs to have exposure to each of these methods.

Division Algorithms

(1) Subtracting the divisor singly.

$32 \div 4$

$$
\begin{array}{rl}
32 & \\
\underline{-4} & 1 \\
28 & \\
\underline{-4} & 1 \\
24 & \\
\underline{-4} & 1 \\
20 & \\
\underline{-4} & 1 \\
16 & \\
\underline{-4} & 1 \\
12 & \\
\underline{-4} & 1 \\
8 & \\
\underline{-4} & 1 \\
4 & \\
\underline{-4} & 1 \\
0 &
\end{array}
$$

This illustrates that there are 8 fours in 32 or $8 \times 4 = 32$, therefore, $32 \div 4 = 8$

(2) For larger numbers it would be impractical to subtract the divisor singly. The process may be shortened by using multiples of the divisor.

$256 \div 4$

$$
\begin{array}{rrr}
4 \,) & \overline{256} & \\
 & \underline{-40} & 10 \\
 & 216 & \\
 & \underline{-80} & 20 \\
 & 136 & \\
 & \underline{-80} & 20 \\
 & 56 & \\
 & \underline{-40} & 10 \\
 & 16 & \\
 & \underline{-16} & \underline{4} \\
 & 0 & 64
\end{array}
$$

This illustrates that there are 64 fours in 256 or $64 \times 4 = 256$; therefore, $256 \div 4 = 64$.

Another method of writing the preceding problem is

```
      4
     10
     20    64
     20
     10
 4 ) 256
    -40
    216
    -80
    136
    -80
     56
    -40
     16
    -16
      0
```

(3) Subtracting multiples of the divisor which are products of 4 and powers of 10.

```
4 ) 1296
    -400     100
     896
    -400     100
     496
    -400     100
      96
     -40      10
      56
     -40      10
      16
     -16       4
       0     324
```

As shown in the preceding example, there is an obvious advantage in subtracting multiples of the divisor which are large block multiples of tens, hundreds, etc.

A further refinement of the preceding method would be to choose the largest multiples of the divisor and powers of ten. To do so will give meaning to the traditional algorithm.

```
4 ) 1296
   -1200     300
      96
     -80      20
      16
     -16       4
       0     324
```

(4) The traditional algorithm

$$\begin{array}{r|llll}
 & \text{Th} & \text{H} & \text{T} & \text{U} \\
 & & 3 & 2 & 4 \\
\hline
4 \,) & 1 & 2 & 9 & 6 \\
 & 1 & 2 & & \\
\hline
 & & & 9 & \\
 & & & 8 & \\
\hline
 & & & 1 & 6 \\
 & & & 1 & 6 \\
\hline
 & & & & 0
\end{array}$$

The letters representing thousands, hundreds, tens and ones placed above the quotient should help the pupil to realize that the 3 in the quotient means that there are 300 fours in 1296, the 2 means 20 fours in 96, etc. These letters should help him to realize htat as he writes the 8 under the T that he is using a place value location allowing him to omit the 0.

Some pupils may prefer to write the zeros. The teacher should allow them to write the zeros if they find it helpful.

A. Incorrect

$$\begin{array}{r}
86 \\
7\,\overline{)\,622} \\
\underline{56} \\
42 \\
\underline{42} \\
0
\end{array}$$

Correct

$$\begin{array}{r}
88 \\
7\,\overline{)\,622} \\
\underline{56} \\
62 \\
\underline{56} \\
6
\end{array}$$

As an operation, division should be presented as the inverse of multiplication. See the strand entitled Operations, Properties and Number Theory. The algorithm generally used in processing division is called the long division algorithm. In example A., the error is actually one of computing differences. More work with processing subtraction problem should correct this kind of difficulty.

B. Incorrect

$46 \div 7 = 6$

Correct

$46 \div 7$ does not name a whole number. Instead, $46 = (7 \times 6) + 4$

Other difficulties may be related to not knowing subtraction or multiplication facts or to errors in processing multiplication and can be remedied by reteaching and practice in related skills. However, the errors in examples B. through E. represent misunderstandings of a deeper nature, that is, misunderstandings of the questions about numbers which the process is designed to answer. To answer the question, "Does 7 divide 46," one asks the related question, "Is there a number which multiplies by 7 such that their product is 46?" Recall of the 7-facts or examinations of the multiplication chart reveals that there is no such number, and one concludes that 7 does *not* divide 46. That is, the mathematical expression "46 ÷ 7" does not name a whole number. However, in application to the real world one is often concerned not so much with the question, "Does one number divide another?," but with two questions - (1) How many subsets of a specified number of objects can be removed from a given set? and (2) How many objects

C. Incorrect

$$\begin{array}{r} 218 \\ 4\,\overline{)152} \\ \underline{8} \\ 7 \\ \underline{4} \\ 32 \\ \underline{32} \\ 0 \end{array}$$

C. Correct

$$\begin{array}{r} 38 \\ 4\,\overline{)152} \\ \underline{12} \\ 32 \\ \underline{32} \\ 0 \end{array}$$

D. Incorrect

$$\begin{array}{r} 2 \\ 42\,\overline{)849} \\ \underline{84} \\ 9 \end{array}$$

D. Correct

$$\begin{array}{r} 20 \\ 42\,\overline{)849} \\ \underline{84} \\ 9 \end{array}$$

E. Incorrect

$$\begin{array}{r} 23 \\ 9\,\overline{)1827} \\ \underline{18} \\ 27 \\ \underline{27} \\ 0 \end{array}$$

E. Correct

$$\begin{array}{r} 203 \\ 9\,\overline{)1827} \\ \underline{18} \\ 27 \\ \underline{27} \\ 0 \end{array}$$

remain? For example, (1) How many subsets of 7 objects each can be removed from a set of 46 objects and (2) How many objects are left over? The process employed in finding these answers is long division. In the process of answering the questions one must remember to find the *greatest* number of subsets which can be removed from the given set.

In example C., 218 subsets of size 4 cannot be removed from a set of 152 objects. In example D., it is true that 2 subsets of size 42 can be removed from a set of 849 objects, but 2 is not the greatest number of such subsets. Also, in example E., 23 is not the greatest number of subsets of size 9 that can be removed from a set of 1827. In each of these three cases, if the pupil had used estimation or intelligent guessing in predicting an approximate number of subsets to be removed, he would have realized that the answer to the first question is not correct.

Note that the long division process can be used to determine whether or not one number divides another. For example, the expression, $5632 \div 176$, names a whole number if and only if there is a single whole number which multiplies by 176 such that their product is 5632, One can instead ask, (1) How many times can 176 be subtracted from 5632? and (2) How many remain? If the answer to question (2) is zero, then we can say that $5632 \div 176 = 32$. The related number sentence is $5632 = 176 \times 32$.

In summary, there are five types of difficulties associated with using the long division algorithm.

(1) Skill in estimation – The development of this skill should begin prior to the introduction of the division algorithm. Estimation skills can be developed in the context of questions such as, "About how many

times can a particular whole number be subtracted from a second whole number?" or "What is a multiple of the number called the 'divisor' which is close to but less than the number called the 'dividend?' Close to but greater than the number called the 'Dividend?' Do you think the answer to the first question is somewhere in between?"

(2) There are *two* answers in the long division process. That is, the process is designed to find answers to two questions.

(3) Division is impossible for many pairs of numbers. For instance, the expressions $622 \div 7$, $46 \div 7$, $849 \div 42$ do not name single whole numbers. In using the long division process, children should realize that if the process yields a non-zero remainder, one concludes that the dividend is not a multiple of the divisor. However, $152 \div 4$ and $1827 \div 9$ do name single whole numbers since the long division process yields zero remainders.

(4) The long division process is not a singular one. It is a combination of processes involving estimation, multiplication, subtraction and addition.

(5) Children, accustomed to memorizing multiplication and addition facts, find it more difficult to think in terms of division and what the operation of division means. See the strand on Operations, Their Properties and Number Theory.

III. Addition of Fractions

	Incorrect	Correct
A.	$\frac{2}{3} = \frac{4}{12}$	$\frac{2}{3} = \frac{8}{12}$
B.	$2\frac{5}{3} = 1\frac{2}{3}$	$2\frac{5}{3} = 3\frac{2}{3}$
C.	$\frac{1}{2} + \frac{2}{3} = \frac{3}{5}$	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$

The causes of errors in addition of fractions may be due to errors in writing sets of equivalent fractions such as examples A. and B. Suggestions for teaching equivalent fractions may be found in the strands, Sets, Numbers and Numeration and Relations and Functions.

Suggestions for teaching addition of fractions may be found in Operations, Their Properties and Number Theory.

IV. Subtraction of Fractions

A. Incorrect

$7\frac{1}{8} - 2\frac{7}{8}$

$$\begin{array}{r} 6\frac{11}{8} \\ -2\frac{7}{8} \\ \hline 4\frac{4}{8} \end{array}$$

Correct

$$\begin{array}{r} 7\frac{1}{8} = 6\frac{9}{8} \\ -2\frac{7}{8} = 2\frac{7}{8} \\ \hline 4\frac{2}{8} = 4\frac{1}{4} \end{array}$$

This error is a carry over from the subtraction of whole numbers. In such computation the pupil would write 1 beside the numeral in the ones place to represent the 1 ten which he had borrowed. Writing it in this manner was all right in computing with whole numbers since he was computing in base ten. However, in adding fractions this pupil did not understand that adding 1 to $\frac{1}{8}$ in this problem actually results in adding $\frac{8}{8}$ to $\frac{1}{8}$.

B.

Incorrect:

$$\begin{array}{r} 7 \\ -5\frac{1}{4} \\ \hline 2\frac{1}{4} \end{array}$$

Correct:

$$\begin{array}{rcr} 7 & = & 6\frac{4}{4} \\ -5\frac{1}{4} & = & 5\frac{1}{4} \\ \hline & & 1\frac{3}{4} \end{array}$$

C.

Incorrect:

$$\begin{array}{r} 3\frac{2}{5} \\ -1\frac{1}{3} \\ \hline 2\frac{1}{2} \end{array}$$

Correct:

$$\begin{array}{rcr} 3\frac{2}{5} & = & 3\frac{6}{15} \\ -1\frac{1}{3} & = & 1\frac{5}{15} \\ \hline & & 2\frac{1}{15} \end{array}$$

Stress the important fact that 1 has many equivalents, and that the pupil should choose whatever form of 1 is needed to give a common denominator in the particular problem. For example, $1 = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5}$ . . . Notice in examples A. and B. the equivalent represent $\frac{8}{8}$ to $\frac{4}{4}$.

The pupil making error B. does not realize that the number "seven" should be represented by symbols other than 7.

It is obvious that the pupil making error C. does not realize that he must have a common denominator when subtracting fractions. See the strand on Operations, Their Properties and Number Theory for a discussion on subtraction of fractions.

V. Multiplication of Fractions

A.

Incorrect: $\frac{2}{3} \times \frac{3}{4} = \frac{8}{9}$

Correct: $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$ or $\frac{1}{2}$

This error is usually made after the student has learned to check for equivalent fractions or after he has studied proportion; in each case he is cross multiplying. Bring to his attention that in working with equivalent fractions the symbol "=" is used, whereas, if fractions are to be multiplied the " X " is used in such problems as A.

B.

Incorrect: $3\frac{2}{5} \times 2\frac{2}{3} = 6\frac{4}{15}$

Correct:

(1)

$$3\frac{2}{5} \times 2\frac{2}{3} =$$

$$(3 + \frac{2}{5}) \times (2 + \frac{2}{3}) =$$

$$6 + \frac{4}{5} + \frac{6}{8} + \frac{4}{15} =$$

$$6 + \frac{12}{15} + 2 + \frac{4}{15} =$$

$$6 + \frac{12}{15} + 2 + \frac{4}{15} =$$

$$9\frac{1}{15}$$

The student may realize that $3\frac{2}{5}$ means $3 + \frac{2}{5}$ but he does not recognize the significance of this principle in this particular exercise. The student may use two different methods to solve this problem.

In (1) the sum of the partial products are given. It might be helpful for the student to compare this method to the partial products as used in multiplying whole numbers such as 23×45 where the sum of the partial products would be $(5 \times 3) + (5 \times 20) + (40 \times 20)$ or $15 + 60 + 120 + 800$ or 995.

(2) $3\frac{2}{5} \times 2\frac{2}{3} =$

$\frac{17}{5} \times \frac{8}{3} =$

$\frac{136}{15} =$

$9\frac{1}{15}$

In (2) each mixed numeral is written and used as one fraction.

To record the numbers used in this method he should realize that $3\frac{2}{5}$ means $3 + \frac{2}{5}$ Therefore, $\frac{15}{5} + \frac{2}{5} = \frac{17}{5}$. The teacher should insist that the pupil use this procedure before using the short cut $3\frac{2}{5}$ is $\frac{5 \times 3 + 2}{5}$ or $\frac{17}{5}$.

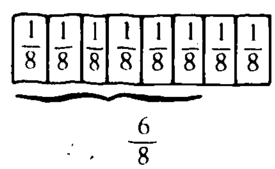
If he uses the short cut with no understanding, he will in later grades make such mistakes as $3\frac{2}{5}$ is $\frac{2 \times 3 + 5}{5}$ or 11.

VI. Division of Fraction

Incorrect	Correct
$\frac{3}{4} \div \frac{2}{3} =$	$\frac{3}{4} \div \frac{2}{3} =$
$\frac{4}{3} \times \frac{2}{3} =$	$\frac{9}{12} \div \frac{8}{12} =$
$\frac{8}{9}$	$9 \div 8 =$
	$1\frac{1}{8}$
	$\frac{3}{4} \div \frac{2}{3} =$
	$(\frac{3}{4} \times \frac{3}{2}) \div (\frac{2}{3} \times \frac{3}{2}) =$
	$\frac{9}{8} \div 1 =$
	$1\frac{1}{8}$

See the Operations, Their Properties and Number Theory strand for a thorough discussion of the operation of division of rational numbers. The two major methods or algorithms for dividing with fractions are illustrated. The common denominator algorithm used in the first correct example can be justified by reading $\frac{9}{12} \div \frac{8}{12}$ as "9 twelfths divided by 8 twelfths," i.e. using the fractional parts, twelfths, as denominations.

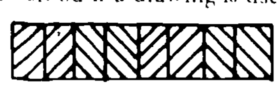
The reciprocal algorithm is used in the second correct example. By multiplying the dividend and the divisor each with the reciprocal of the divisor, the quotient remains unchanged and the divisor becomes 1. Of the two methods, the reciprocal method is more widely used; however, few pupils understand why it works and often invert the wrong fraction. The common denominator method may not be as well known, but it is more readily understood by children. The common denominator method may be illustrated by using strips as shown. (a) Find the number of eighths in six-eighths or $\frac{6}{8} \div \frac{1}{8} = \square$.



The pupil can easily see there are 6 eighths in six-eighths.

(b) Find the number of one-fourths in six-eighths or $\frac{6}{8} \div \frac{1}{4} = \square$.

The problem written as $\frac{6}{8} \div \frac{1}{4} = \frac{6}{8} \div \frac{2}{8}$ is more readily understood if a drawing is used.



The pupil can see that there are 3 two-eighths or 3 one-fourths in 6 eighths.

VII. Division of Decimals

Incorrect	Correct
$53.95 \div 8.3$	$83.\overline{)53.95}$ quotient 6.5
$8.3\overline{)53.95}$ quotient .65; 498; 41 5; 41 5	

Before beginning computation with decimals, it will be helpful to explain thoroughly that decimal fractions are an extension of base ten place value. The difficulty arises in the present example because the pupil does not understand the extension of place value. Division of decimals can be presented as the inverse of multiplication. In the example shown, 8.3 is a factor and 53.95 is a product. Ask the question, "Tenths $\times \square$ = hundredths or $\frac{1}{10} \times \square = \frac{1}{100}$?"

The pupil should see that 1/10 is a replacement for the $\square$, and that the missing factor would be one tenth. One way of showing this method is as follows.

$$83 \text{ tenths}\overline{)\,5395 \text{ hundredths}} \quad \text{quotient: } 65 \text{ tenths or } 6.5$$

A second method is to record the problem using common fractions so that the divisor and divident have common denominators.

$$53.95 \div 8.3 = \frac{5395}{100} \div \frac{83}{10}$$

$$= \frac{5395}{100} \div \frac{830}{100}$$

A third method is to record the problem as one common fraction and multiply the numerator and denominator by a power of ten in order to make the denominator (divisor) a natural number.

$$\frac{53.95}{8.3} = \frac{53.95}{8.3} \times \frac{10}{10} = \frac{539.5}{83}$$

The third method is the one usually used with carets placed to show the placing of the decimals. If this method is used, care must be taken in recording to align the decimal of the quotient with the caret in the dividend.

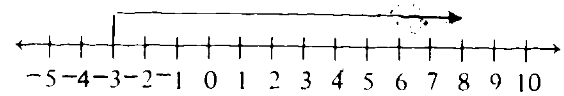
VIII. Subtraction of Integers

	Incorrect	Correct
A.	$^{+}8 - {}^{-}3 = 5$	$^{+}8 - {}^{-}3 = {}^{+}11$
B.	$^{+}8 - {}^{-}3 = {}^{-}11$	$^{+}8 - {}^{-}3 = {}^{+}11$

Errors in subtraction of integers are from two causes. (1) The pupil does not understand the operation with integers, or (2) he makes mistakes using combinations of whole numbers.

To alleviate errors due to the first cause consider subtraction of integers as the inverse of addition. In this problem, $^{+}8 - {}^{-}3$ may be stated as, "What number added to negative three will give a sum of eight?" (Note that $^{-}3$ is read "negative three" *not* "minus three." "Minus" is reserved for the subtraction symbol.

This problem of subtraction of integers, $^{-}3 + \square = {}^{+}8$ can be illustrated with a number line.



The points corresponding to the known addend and the sum are marked on the number line. The pupil should then count the number of units from the addend to the sum to find the missing addend. If he counts toward the right, the missing addend is a positive number; and if he counts to the left, the missing addend is negative. The pupil should have experiences using the number line to solve different types of subtraction problems with integers.

The pupil should be encouraged to observe many examples in order to discover the pattern that subtracting an integer gives the same result as adding its opposite. The teacher should provide many examples with correct answers.

They must be selected carefully in order to have the patterns necessary to guide discovery. One set of examples to use is as follows.

$^{+}8 - {}^{-}5 = \boxed{^{+}3}$	$^{-}8 + {}^{-}5 = \boxed{^{-}13}$
$^{+}8 + {}^{-}5 = \boxed{^{+}3}$	$^{+}4 - {}^{+}6 = \boxed{^{-}2}$
$^{-}8 - {}^{-}5 = \boxed{^{-}3}$	$^{+}4 + {}^{-}6 = \boxed{^{-}2}$
$^{-}8 + {}^{+}5 = \boxed{^{-}3}$	$^{-}4 - {}^{-}6 = \boxed{^{+}2}$
$^{-}8 - {}^{+}5 = \boxed{^{-}13}$	$^{-}4 + {}^{+}6 = \boxed{^{+}2}$

IX. Multiplication of Integers

Incorrect	Correct
$8 \times 5 = 40$	$8 \times 5 = {}^{+}40$

Errors in multiplication of integers are from two causes. (1) The pupil does not know or understand the definition of the operation, or (2) he makes mistakes in multiplying the related whole numbers.

The definition of multiplication for integers is more difficult for pupils to understand because it is more difficult to relate the operations to objects or physical situations. Of course, a negative number times a negative number is most difficult. There are many methods of justifying the answer for this problem. The pupils should be exposed to several so that he may choose the one he believes. The physical situations may seem artificial to some students, but there are several that can be used. See those used in the strand on Operations, Their Properties and Number Theory.

A more mature student will enjoy a simple proof using several important properties for integers such as $a \times 0 = 0$, $a \times 1 = a$, $a \times {}^-a$. For example, if a student has a problem $^-6 \times {}^-9 = \square$ he may be able to understand a proof of multiplication of two specified integers as the one below although he may not be able to produce it.

Statement	Reason
$^+9 + {}^-9 = 0$	Inverse element for addition of integers
$^-6(^+9 + {}^-9) = 0$	$a \times 0 = 0$
$^-6 \times {}^+9 + {}^-6 \times {}^-9 = 0$	distributive property
$^-54 + {}^-6 \times {}^-9 = 0$	multiplication facts
$^-54 + {}^+54 = 0$	additive inverse
Therefore	
$^-6 \times {}^-9 = {}^+54$	The additive inverse of a number must be unique.

An advanced student may want to do the proof for the general case $^-a \times {}^-b = {}^+ab$.

UPDATING CURRICULUM

Continuing Program Improvement
Evaluating Pupil Progress
Utilization of Media

CONTINUING PROGRAM IMPROVEMENT

Long range curriculum planning is necessary for the development of a good mathematics program. A curriculum committee should be in continuous operation to evaluate, to improve or to revise the program. The committee should schedule evaluations of the program for specified times; this schedule should be correlated with the schedule of evaluation of the pupils' progress.

The plan of the local program should be in writing and should be available to all of the elementary teachers. This guide is not a course of study for the local schools since it is not all inclusive. It should serve as a guideline in planning the mathematics program of the local elementary schools. The objectives in the local guide should be an expansion of those in the state guide. Ideally, the objectives at the local level should be written in specific, measurable, behavioral terms. The objectives should be determined before the textbooks and other materials are selected since the textbook is not the course of study but is one of the tools to aid instruction.

The mathematics program should

- be consistent with the total program of the local system so that there will be correlation with other curriculum areas;
- fulfill the objectives as presented in the local mathematics guide;
- be an expansion of the state mathematics guide;
- have consistent, accurate and precise vocabulary;
- include plans for best use of textbooks and other materials;
- include plans for best use of audio-visual aids and ETV;
- provide for continuous improvement;
- reflect research findings of recognized authorities in mathematics education.

Writing Local Guides

The local guides do not need to follow one pattern. They should be in accordance with local needs and should be in a form for best utilization. Some guides may be an enlargement on or extension of special units specifically selected by the local staff.

In order to achieve the goals of the mathematics curriculum it is essential to have well planned, regularly scheduled inservice programs. In these meetings, the staff of the local schools should utilize the state guide. The inservice program could center around one or more of the following.

- stating specific objectives, each one written in terms of the desired behavior, the situation involving such behavior and the criteria for success;
- writing additional activities in each of the strands or in specific strands;
- having indepth studies in one or more strands so as to develop them further;
- developing a mathematics laboratory and writing plans for its use considering effective staff utilization, organization of materials and facilitation of their use and instructions to students through such devices as activity cards;
- collecting and organizing various materials to use in teaching the strands;
- developing local guides to fit the local needs, such as to develop specific skills needed by local industries;
- developing charts for recording the evaluation of pupil progress.

Achievement of Goals

Pupil achievement should be in terms of growth, change and progress in the attainment of locally established objectives. Ability to use many processes in appropriate situations is an important facet of achievement. The mathematics curriculum should provide continuous opportunity for the pupil to progress in a program at his level of ability. The teacher has a sense of achievement if he has provided an opportunity for success in the program for his pupils.

The program should be flexible enough to encourage innovative practices such as trying out new materials and new methods. *Ideas for these may be obtained through participating in professional meetings, reading and studying professional journals and other publications and working with consultants and other resource people.*

The excellence of a program is perhaps best represented by the manner in which a suitable balance is achieved between challenge to the student on one hand and opportunity for success on the other.

Selection of Textbooks

Since there is multiple adoption of textbooks for the state public schools, the local systems need to establish criteria for selecting those which will be locally adopted. It is suggested that a local committee be composed of representatives from each school and from the different grade levels. This committee should begin its work one year prior to the selection of textbooks at the local level. It should study local needs considering both the experiences of the pupils and the background of the teachers.

When the committee has narrowed its selections to a few books, pilot programs should be instituted in representative cases where financially feasible and comparisons made of the results found before the committee makes the final selection. This practice would alleviate the dangers of having individual teachers making the choice of textbooks and of having no sequence throughout the school program.

The textbook committee should accept as its responsibility the selection of diversified textbooks for different levels to suit individual needs. Also, textbooks for supplemental uses could be chosen as well as those for experimental purposes.

If textbook money is limited, the committee should investigate sources of revenue for implementing the needs, such as the various federal funds. If sufficient funds are not available to purchase basic textbooks for all levels, a careful study should be made to determine the best usage of funds.

In selecting textbooks, the local committee should consider specific criteria. Among publications in which such listings may be found is the *Criteria for Selecting Textbooks* of the National Council of Teachers of Mathematics. Criteria such as the following should be taken into account.

- Throughout the series does the content include sufficient material to develop the concepts as stated in the objectives of the strands in the state mathematics guide?
- Is the spiral approach to learning emphasized in the content? Are topics explored in more depth at each level?
- Is the method of presentation conducive to logical thinking through pupil involvement and discovery of patterns and principles of mathematics?
- Does the presentation of material allow a child to move ahead at his own rate and interest level?
- Is the material appropriate for the child's vocabulary and reading level?
- Is the arrangement of material adaptable to all levels of ability?
- Is the review of previous level material distributed throughout the content areas rather than concentrated in the beginning of the book?
- Is the drill work designed to reinforce the basic concepts?
- In the series, is there multiple authorship which includes a mathematician and a teacher experienced at this level?
- Are there sufficient diagnostic and evaluative materials, including those for self-evaluation? (Such materials could be included in text, workbook or manual.)
- Do the teacher aids include a rationale, the pupil's material, suggestions for individualizing instruction, additional resources and evaluation procedures, and are these aids convenient to insure effective use?
- Are the textbooks durable, attractive, of convenient size, of good quality material, of suitable type and functionally arranged?

EVALUATING PUPIL PROGRESS

Evaluation is diagnostic, prescriptive and individualized, and should facilitate self-evaluation. The evaluation is developed by many methods and from many sources. Evaluation may be made in the context of projects, problem-solving situations, daily class participation, paper and pencil tests and standardized tests given for diagnostic purposes. For advice on selection of tests for specific purposes consult with the Guidance and Testing Unit of the Georgia Department of Education.

Projects should challenge the pupil without frustrating him, provide opportunity for applying mathematical concepts already learned, be appropriate to the interest and ability level of the individual and be shared after completion. Problems should arise naturally from classroom situations. A teacher may contrive problems to relate concepts learned to real situations and to challenge the pupil to reason and think beyond the level at which he has computational competency. Class participation should involve every child, build confidence in the individual, create an atmosphere conducive to building a positive self-image and should not be dominated by any one individual.

Daily performance of pupils including questions pupils ask and their response to teacher queations are excellent ways to evaluate the understanding of the pupils. Activities which allow pupils to demonstrate understanding of specific objectives are good instruments through which to measure understanding.

Tests should be given to determine pupil achievement of specific objectives, and can be varied by requiring descriptions of ways to solve specified problems rather than requiring numerical answers.

Teachers need to make tests that are short enough to give the pupil time to think and complete the test in the time alloted. Teachers need to be careful to give specific and clear instructions. Tests may be used as learning experiences when they are returned and discussed with pupils. Tests should be well balanced to cover the objectives which are to be tested. Tests should be checked and returned so that the pupil gets immediate feedback.

Evaluation procedures should provide the pupil with an understanding of his progress and difficulties. Evaluations should be made often enough to direct the development of the curriculum for the individual. Records should be kept in a form which clearly shows the progress of the pupil.

The charts given here are suggestions and only one strand has been used for illustrative purposes. The teacher may wish to sub-divide the larger topics into specific concepts which can be taught in short time periods. The process of developing his own charts will help the teacher to determine the direction in which he plans to move and assess what he has accomplished when he checks the progress of the pupils.

It is most important that the receiving teacher know at what level the pupil was performing at the end of the previous school year or at the time of transfer. For this reason, a cumulative record should be kept for each child. This chart may be checked by recording at the end of the presentation of a strand or at the end of the school year. To facilitate the record keeping, the teacher will probably prefer to keep a class record and then transfer the information to individual records. By seeing the class record as a whole, the teacher can see the strengths and weaknesses of the total class.

Records of individual pupil's progress should be kept in his cumulative folder and passed on to the next teacher. A chart should be made for each strand. (See Chart 1 Pupil Progress.) Charts should be made on $8\frac{1}{2}''$ by $11''$ paper for easy filing.

There are three levels of progress indicated for each child. The first level indicates only that the teacher has presented this concept to the child. The second level indicates that the child uses the concept under directions. The third level indicates that the child uses the concept independently.

SAMPLE

Individual Pupil Progress Sheet
Sets, Numbers, Numeration
Upper Elementary Grades

Name ______________________

Date ______________________

Grade ______________________

OBJECTIVES	Teacher has introduced	Pupil shows understanding with guidance	Pupil can accomplish independently
1. Tabulate and/or describe sets			
2. Pick out from a given set, subsets having a specified common property			
3. Identify common properties of a given set			
4. Use the language of sets to describe and organize information			
5. Read and write large numbers utilizing the period notation			
6. Translate large numbers into expanded exponential form			
7. Demonstrate place value by using another system of notation			
8. Identify so-called figurate number, i.e. square numbers, triangular numbers and rectangular numbers			
9. Use divisibility tests for various numbers and casting out nines			
10. Name the ordered pair of counting numbers associated with fractional parts of (a) units (b) sets			
11. Discriminate between an ordered pair of whole numbers used in a rate context and an ordered pair of whole numbers used in a fraction context			
12. Name equivalent fractions (ordered number pairs) associated with equivalent partitionings of (a) units (b) sets			
13. Generate a finite number of members of the set of equivalent fractions to which a given fraction belongs			
14. Determine if two ordered number pairs are equivalent to each other (a) by inspection of sets of equivalent number pairs (b) by using the test for equivalence			
15. Identify the point on the number line with which a given set of equivalent fractions is associated			
16. Partition a set of fractions into mutually exclusive subsets according to whether each fraction (a) is equivalent to a decimal fraction (b) is not equivalent to a decimal fraction			
17. Record fractions (a) in basic fraction form (b) in mixed numeral form (c) in decimal notation			
18. Find the equivalent non-terminating decimal notation for any non-decimal fraction using the process of approximations			
19. Find within a specified margin of error, a decimal fraction and write in decimal notation			
20. Name the ordered pair of numbers which expresses the ratio of two disjoint sets or the ratio of a specified subset of a set to the given set			
21. Discriminate among the interpretations of ordered number pairs as used in the context of fractions, of ratios and of quotients of whole numbers			
22. State the definition of rational numbers as an element of a mathematical system			
23. Identify and describe everyday situations that require the use of directed numbers			
24. Construct the set of opposites of the whole numbers and the opposites of the opposites which form the set of integers			
25. Identify the following subsets of the set of integers: set containing zero; set of positive integers; set of negative integers; the set of non-negatives; the set of non-positives			
26. Order any two or more given integers			

Chart 1

DIAGNOSTIC CLASS CHART*

Name of pupil	Sets				Whole Numbers						Rational Numbers										Integers			
	Describe	Tabulate	Recognize common properties	Use set language	Read large numbers	Write large numbers	Use expanded notation	Use exponential form	Use another base	Recognize figurate numbers	Use in rate context	Use in fraction context	Generate sets of equivalent fractions	Match fractions to point on number line	Partition into subsets equivalent to decimals	Not equivalent	Record basic form	Use mixed numeral form	Use decimal notation	Define as element of mathematical system	Describe everyday situations using directed numbers	Construct the set of opposites	Identify subsets of integers	Order integers
1. Adams, Bill																								
2. Brown, Ann																								

Chart 2

*A Diagnostic Class Chart with a sample of objectives from some of the strands is given. A diagnostic chart with objectives for the year will help the teacher see the strengths and weaknesses of the class before beginning a strand. Such a chart could also be used as the teacher completes his presentation of material from each strand.

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TEACHER SELF EVALUATION CHART*

Pupil's name	Recognizes sets and subsets	Recognizes intersection of two sets	Recognizes union of two sets	Distinguishes between finite and infinite sets	Writes and solves number sentences	Uses number lines	Records numbers in various ways	Uses properties in naming numbers	Counts numbers to millions	Writes numbers in exponent form	Counts in base 5	Compares base 10 and base 5	Uses metric system in measuring
1. Bell, Ann													
2. Carter, Sue													

Chart 3

*A teacher may desire to make small learning step analyses of his pupils. This chart is a sample of small steps one teacher selected for her sixth grade.

UTILIZATION OF MEDIA

INTRODUCTION

The use of a variety of materials can improve learning and instruction, since the individual needs of children vary and individual children learn by different means. These materials help to stimulate new ideas and concepts enabling children to solve problems whose solutions are as yet unlearned responses. A variety of media should be readily accessible. In planning and organizing the housing of materials those often used and those occasionally used should be taken into consideration. This variety of media should include materials which are adaptable to each pupil's particular ability level.

Many valuable teaching aids may be made inexpensively, and pupil participation in their construction serve as a learning activity. A number of books are available which assist one in making aids. A number of these aids are listed in the references.

Many worthwhile, commercially prepared aids are now on the market. A special committee composed of those who determine the purpose and method of instruction in the local school should participate in the selection of instructional materials. These instructional materials should be examined or previewed to determine their probable effectiveness in the situation where they are to be used. Instructional materials in kits, packages and series should be thoroughly examined prior to approval of purchase.

Audio-visual equipment can be used in various ways, and teachers should be alert to innovations in the use of all media. For example, the overhead projector may be used in many ways other than for projecting transparencies. In the study of sets, collections of small objects may be placed on the overhead projector and their shadows can easily be seen by all pupils. The tape recorder may be used in work with individuals or small-groups; it is especially helpful with the slow reader.

The Georgia Department of Education television network has regularly scheduled programs in mathematics at all levels of elementary school. These programs supplement and enrich the classroom teacher's presentation. Schedules and guides giving information on each lesson are available from the director of programming.

Films and filmstrips should be previewed and carefully selected before use in the classroom. The Georgia Department of Education publishes an up-to-date list of films and filmstrips available through the Audio-visual Service.

Magazines such as *The Arithmetic Teacher* and *The Mathematics Teacher,* publications of the National Council of Teachers of Mathematics, contain excellent articles for teachers. Many valuable books are available through the Georgia Department of Education Readers' Services of the Public Library Unit and may be requested through the school or public librarian.

The list of materials which follows is not meant to be all-inclusive. These materials should be very useful, and teachers are encouraged to try many of them. The reference lists are limited and are annotated for the use of strands or subjects in the guide. The teacher may find these references useful in strands other than those for which they were annotated as well as references which are not listed.

The instructional aids for use in the activities of the strands are listed.

Teaching practices involving various instructional materials should be carefully evaluated in terms of effective results. The information relative to those materials whose use produces the most favorable results should be shared with teachers not only in the school system but with others. Provisions should be made for frequent revision of any locally approved list of instructional materials to allow for deletions and additions needed to update the curriculum.

Materials on media that are available through the Georgia Department of Education, State Office Building, Atlanta, Georgia 30334, and the offices from which they may be obtained are listed below.

Viewpoints; Instructional Materials – Selection at State and Local Levels, Suggestions for Use

Director of Elementary and Secondary Education

Catalog of Classroom Teaching Films for Georgia Schools

Audio Visual Unit

Selected List of Books for Teachers

Director, Readers' Services, Public Library Unit

Georgia Library List for Elementary and High Schools

Director, School Library Services Unit

In-school television schedules and guides for educational television

Director, Television Programming
Georgia Educational Television
Georgia Department of Education
1540 Stewart Avenue, S. W.
Atlanta, Georgia 30310

REFERENCES

Abbott, Janet S., *Learn to Fold – Fold to Learn.* Chicago: Lyons and Carnahan, 1968.
A pupil workbook about reflections; teacher's edition available also.

____________, *Mirror Magic. Chicago: Lyons and Carnahan, 1968.*
A pupil workbook about reflections and symmetry; teacher's edition available also.

Association of Teachers of Mathematics , *Notes on Mathematics in Primary Schools.* Cambridge University Press, 1968.
Suggestions and lessons written by teachers for primary teachers use. This book is available from Cuisenaire Company of America, Inc., New Rochelle, New York.

Banks, J. Houston , *Learning and Teaching Arithmetic.* Boston: Allyn and Bacon, Inc., 1960.
Several chapters have suggestions related to difficulties with computation.

Bendick, Jeanne and Levin, Marcia, *Mathematics Illustrated Dictionary.* New York: McGraw-Hill Book Co., 1965.
This book is a student dictionary containing many of the terms as they are used in the guide. It also contains diagrams and pictures helpful to children.

Berger, Melvin, *For Good Measure: The Story of Modern Measurement.* New York: McGraw-Hill, 1969.
Interesting and little-known facts about the development of systems of measurement, the importance of precise measurement in science and industry and the many ways that measurement is used subconsciously.

Bloom, Benjamen, et. al., *Taxonomy of Educational Objectives. Handbook I: The Cognitive Domain.* New York: David McKay Co., 1956.
A help with making and interpreting tests.

Bowers, Henry and Joan, *Arithmetic Excursions: An Enrichment of Elementary Mathematics.* New York: Dover Publications, Inc., 1961.
Chapter 18 contains interesting information about figurate numbers, perfect numbers and amicable numbers.

Brumfiel, C. F., Eicholz, R. E., and Shanks, M. E., *Fundamental Concepts of Elementary Mathematics.* Reading, Massachusetts: Addison-Wesley Co., Inc., 1962.
A book in mathematics for teachers; provides background material for some concepts of geometry and other concepts of the guide.

Buros, O. K., *The Sixth Mental Measurements Yearbook.* Highland Park, New Jersey: Gryphon Press, 1965.
A help with making and interpreting tests.

Cambridge Conference, The, *Goals for the Correlation of Elementary Science and Mathematics.* Boston: Houghton Mifflin Co., 1969.
This book for teachers includes the development of relations and functions. The importance of application of the equivalence relation is emphasized.

Copeland, Richard W., *How Children Learn Mathematics.* New York: The Macmillan Co., 1970.
This book emphasizes how children learn mathematics, rather than techniques of teaching elementary school mathematics. The role of the teacher is suggested to be that of a skillful interviewer. Illustrations show the teacher how to use laboratory or manipulative materials which help the child learn mathematical concepts at a concrete operational level.

D'Augustine, Charles H., *Multiple Methods of Teaching Mathematics in the Elementary School.* New York: Harper and Row, 1968.

The book involves prospective teachers in the creation of exercises that can lead children to make discoveries of various number patterns and problems to lead to mathematical discovery. Many methods of presenting and using principles are given.

Davis, Robert B., *Explorations in Mathematics.* Palo Alto: Addison-Wesley Publishing Co., 1966.

Reference for the teacher. This book is especially helpful in providing activities on functions.

Dienes, Zolton P. and Golding, E. W., *Exploration of Space and Practical Measurement.* New York: Herder and Herder, 1966.

This book is a teacher's guide for developing geometry and measurement in the lower grades. Children are introduced to these topics by means of games of reflections, turning and measuring by using arbitrary units and later standard units.

__________, *Geometry of Congruence.* New York: Herder and Herder, 1967.

A booklet for teachers describing activities for pupils on reflections, rotations and translations.

__________, *Geometry of Distortion.* New York: Herder and Herder, 1967.

A book for teachers describing activities for pupils on topological equivalence and stretches and shrinkage.

__________, *Groups and Coordinates.* New York: Herder and Herder, 1967.

For teachers; describes activities for graphs.

__________, *Learning Logic; Logical Games.* New York: Herder and Herder, 1966.

This book gives the teacher, through narrative diagrams, directions for pupil activities with attribute blocks.

__________. *Sets, Numbers, and Powers.* New York: Herder and Herder, 1966.

A reference for the teacher; this booklet contains activities leading to an understanding of sets and numbers.

Duncan, Ernest G. and Quast, W. G., *Modern School Mathematics Workbook for Elementary Teachers.* Boston: Houghton Mifflin, 1968.

Provides a thorough treatise of mathematics needed by the elementary teacher through explanation, questioning and drill.

Ebel, R. L., *Measuring Educational Achievement.* Englewood Cliffs, New Jersey: Prentice-Hall, 1965.

A help with making and interpreting tests.

Elementary Science Study, Educational Development Center, *Attribute Games and Problems.* St. Louis: Webster Division, McGraw-Hill Co., 1968.

A variety of materials with teacher's guide for developing skills in problem solving, especially developing skills in classifying and setting up relationships between the classes.

Fitzgerald, William, et. al., *Laboratory Manual for Elementary Mathematics.* Boston: Prindle, Weber and Schmidt Inc., 1970.

An excellent reference for the teacher; this manual establishes a discovery approach for elementary teachers to find solutions to problems using many mathematical manipulative materials.

Ford Motor Company, *History of Measurement.* Dearborn, Mich.: Research Division.

This booklet gives historical development of measurement.

General Motors, *Precision A Measure of Progress.* Detroit, Michigan: General Motors Corporation, 1952.

This Booklet gives historical development of precision of measurement.

Glennon, Vincent J., and Callahan, Leroy G., *Guide to Current Research, Elementary School Mathematics*. Washington: Association for Supervision and Curriculum Development, NEA, 1968.

Gives the teacher a ready reference to research pertinent to the field of mathematics as applied to elementary schools.

Grossnickle, Foster E., Brueckner, Leo J. and Reckzeh, John, *Discovering Meanings in Elementary School Mathematics*. Fifth edition. New York: Holt, Rinehart and Winston, Inc., 1968.

Illustrates how the basic principles of learning are applied in presenting a given topic. Great stress on structure as the dominant theme in elementary mathematics.

Hasford, Philip L., *Algebra for Elementary Teachers*. New York: Harcourt, Brace and World, Inc., 1968.

This book helps the elementary teacher understand operations in algebraic terminology.

Heinke, Clarence, *Fundamental Concepts of Elementary Mathematics*. Encino, Cal.: Dickinson Publishing Co., Inc., 1970.

The chapter entitled "Algorithms of Elementary Arithmetic" has helpful suggestions on computation.

Hoghen, Lancelot, *The Wonderful World of Mathematics*. Garden City, N. Y.: Doubleday, 1955.

The development of mathematics through the ages is described in story and pictures.

Hartung, Maurice L., and Walch, Ray, *Geometry for Elementary Teachers*. Glenview, Illinois: Scott, Foresman and Co., 1970.

A book for teachers which explains certain geometric relations; the basic constructions; and reflections, rotations, translations and stretches.

Horne, Sylvia, *Learning About Measurement*. Chicago: Lyons and Carnahan, 1968.

A book of student activities in measurement.

Huff, Darrell, *How to Lie with Statistics*. New York: W. W. Norton and Co., Inc., 1954.

This book could be used as interesting reference material for the teacher.

Johnson, Donovan A., Glenn, William H. and Norton, M. Scott, *Exploring Mathematics on Your Own*. St. Louis, Missouri: Webster Publishing Co., 1960.

These 18 booklets are readable sources for teachers. There are directions for numerous activities which illustrate specified relations. The booklets may be purchased separately. Some of the titles are *Topology – The Rubber Sheet Geometry, The World of Measurement, Curves in Space, Pythagorean Theorem, Geometric Constructions, Probability and Chance, The World of Statistics.*

Johnson, Donovan A. and Rising, Gerald R., *Guidelines for Teaching Mathematics*. Belmont, Cal.: Wadsworth Publishing Co., Inc., 1967.

The chapter "Developing Computational Skills," has suggestions for overcoming difficulties in computation, and the chapter, "Evaluation of Achievement," has suggestions on various types of evaluation. Other chapters deal with basic techniques and materials. An excellent sourcebook for the teachers.

Kennedy, Leonard M., *Models for Mathematics in the Elementary School*. Belmont, Cal.: Wadsworth Publishing Co., Inc., 1967.

This book has descriptions of many aids to make and use in teaching different topics in elementary mathematics.

Linn, Charles F., *Puzzles, Patterns, and Pastimes from the World of Mathematics*. Garden City, N. Y.: Doubleday, 1969.

Puzzles and mathematical games, both ancient and modern, to test the skill of the reader and to stimulate him to invent similar ones.

Mann, William et. al., *Measures*. Columbus, Ohio: Charles E. Merrill Publishing Co., 1968.

This booklet is one of the *Independent Learning Series*. It includes historical development in measurement and exercises for pupils.

Marks, John L., Purdy, Richard and Kinney, Lucien B., *Teaching Elementary School Mathematics for Understanding.* Second Edition. New York: McGraw-Hill Book Co., 1965.

Chapter six has some suggestions for techniques for fixing skills.

Merton, Elda L., and May, Lola June, *Mathematics Background for the Primary Teacher.* Wilmette, Ill.: John Colburn Associates, Inc., 1966.

A reference for teachers of K-3. The presentation is in the form of charts with explanations of eighteen topics taught at the primary level.

Mueller, Francis J., *Arithmetic, Its Structure and Concepts.* Second Edition. Englewood Cliffs, N. J.: Prentice Hall Inc., 1964.

Chapters two and three have extensive discussions of direct operations and inverse operations.

National Aeronautics and Space Administration and U. S. Office of Education, *What's Up There;* Teachers' Edition. Washington, D. C.: U. S. Government Printing Office, 1964.

A source book in space oriented mathematics for grades five through eight.

National Council of Teachers of Mathematics, 1201 Sixteenth Street, NW, Washington, D. C., 20036

Aids for Evaluators of Mathematics Textbooks, 1965.

A set of criteria to aid elementary and secondary teachers in selecting textbooks; pamphlet.

Boxes, Squares, and Other Things: A Teacher's Guide for a Unit in Informal Geometry.

Experiences for pupils in visualizing objects and in the concepts of transformations and symmetry.

Evaluation in Mathematics. Twenty-sixth Yearbook, 1961.

A discussion of and suggestions for evaluation of instruction.

Enrichment Mathematics. Twenty-seventh Yearbook, Second Edition, 1963.

Very brief discussion of topics pertinent to elementary school including rationale and appropriate activities.

Experiences in Mathematical Ideas, Vol. 1, 1970.

Designed to help teachers stimulate slow learners in grades 5-8. This project combines a text for teacher use with laboratory oriented package including loose-leaf materials to be duplicated for student work-sheets.

Formulas, Graphs, and Patterns: Experiences in Mathematical Discovery, 1, 1966.

Describes activities for pupils; pamphlet.

Geometry: Experiences in Mathematical Discovery, 4, 1966.

Describes activities for pupils; pamphlet.

Growth of Mathematical Ideas, Grades K-12. Twenty-fourth Yearbook, 1959.

A general survey of mathematics curriculum including sections on "Ratio-like Numbers," "Fractions as Ordered Pairs of Numbers" and "Language and Symbols."

Instruction in Arithmetic. Twenty-fifth Yearbook, 1961.

Includes suggestions on computation. Provides background for teaching any modern elementary school mathematics program.

Mathematics Library—Elementary and Junior High School, 1968.

An annotated bibliography of enrichment books for grades K-9, classified by grade level.

More Topics in Mathematics for Elementary School Teachers. Thirtieth Yearbook, 1969.

This book provides background information for teachers on the key principles of mathematics.

Paper Folding for the Mathematics Class, 1957.

Directions for paper folding activities which illustrate selected geometric relations.

Readings in Geometry from "The Arithmetic Teacher," 1970.

A booklet of articles containing suggestions for classroom activities.

Topics in Mathematics for Elementary School Teachers. Twenty-ninth Yearbook, 1964.

Gives the key principles for an understanding of the major topics. One chapter on the rational number system deals with various interpretations of rational numbers as well as the numeral forms— fractional, decimal and mixed numerals used to represent rational numbers. A chapter on sets includes basic ideas on relations. One chapter gives a thorough discussion of number systems for whole numbers and rational numbers.

uffield Mathematics Project, *Teachers' Guides: Computation and Structure, Graphs Leading to Algebra* and *Shape and Size.* New York: John Wiley and Sons, Inc.

The teachers' guides are for elementary mathematics activities designed to encourage children to discover mathematical processes for themselves. The material is written in three main streams stated above. Each stream is written in a number of booklets in the stages of development of children.

Overman, James Robert, *The Teaching of Mathematics.* Chicago: Lyons and Carnahan, 1961.

Chapters 8-15 give suggestions on teaching addition, subtraction, fractions and denominate numbers.

Papy, Frederique, *Graphs and the Child.* Montreal: Algonquin Publishing Co., 1970.

A description of a series of ten lessons on graphs for six-year-olds helps the mathematical notion of relation to emerge.

A description of a series of ten lessons on graphs for six-year-olds helps the mathematical notion of relation to emerge.

Peterson, John A. and Hashisaki, Joseph, *Theory of Arithmetic. New York: John Wiley and Sons, 1967.*

Rational numbers are treated in terms of several interpretations, including both the fractions and rate-pair interpretations, which are appropriate to the elementary school curriculum.

Phillips, Jo M. and Zwoyer, R. E., *Motion Geometry, Book 1: Slides, Flips, and Turn* ew York: Harper and Row, 1969.

A pupil workbook about translations, reflections and otations teacher's edition, containing helpful notes, available. For students in the upper grades.

Riedesel, C. Alan, *Guiding Discovery in Elementary School Mathematics.* Ne York: Appleton-Century-Crofts, 1967.

This book provides prospective and inservice elementa teachers with illustrative situations that make use of modern mathematical content and ideas to develop a guided discovery approach to teaching mathematics in the elementary school.

Sanders, N. M., *Classroom Questions: What Kinds?* New York: Harper and Row, 1966.

A help with making and interpreting tests.

School Mathematics Study Group, A. C. Vroman, Inc., 2085 E. Foothill Boulevard, Pasadena, California, 91109

Factors and Primes, 1965.

A book written for high school students; a good reference for the elementary school teacher. A teacher's commentary is available.

Mathematics for the Elementary School, 1962.

This book for elementary school mathematics includes student exercises in measurement.

Mathematics for Junior High School, Volume I, Parts 1-2; Volume II, Parts 1-2, 1961.

This book for junior high schools includes student exercises in measurement.

Introduction to Probability, Part 1, Basic Concepts, 1967.

This is excellent material that is appropriate for classroom use in grades 1-8.

Introduction to Probability, Part 2, Special Topics, 1967.

This is excellent material appropriate for use in the classroom in the upper elementary grades.

Probability for Primary Grades.

This booklet is for student use. A teacher's commentary and a set of spinners for the classroom are available.

Probability for Intermediate Grades.

This booklet is for student use. A teacher's commentary and a set of spinners for the classroom are available.

Secondary School Mathematics, 1970.

This material is nongraded for low achievers in mathematics in junior high school.

Starr, John U., *The Teaching of Mathematics in the Elementary Schools.* Scranton, Penn.: International Textbook Co., 1969.

This book emphasizes why and how various mathematical principles and concepts operate, and provides teachers with proven and class-tested techniques to help implement learning concepts.

Swain, Robert L., *Understanding Arithmetic.* Revised by Eugene D. Nichols. New York: Holt, Rinehart and Winston, Inc., 1965.

Two chapters give suggestions on computations with directed numbers and the complement method of subtraction. One chapter on number theory includes a discussion on divisibility.

Thorndike, R. L. and Hagen, Elizabeth, *Measurement and Evaluation in Psychology and Education.* New York: John Wiley (Second Edition), 1961.

A help with making and interpreting tests.

Torrance, E. Paul and Myers, R. E., *Creative Learning and Teaching.* Chapters 7, 8. New York: Dodd, Mead, 1970.

A help with making and interpreting tests.

Turner, Ethel M., *Teaching Aids for Elementary Mathematics.* New York: Holt, Rinehart and Winston, Inc., 1966.

This is a source book for teachers, including many activities for students. The activities using coordinates are useful.

Van Engen, Henry, et. al., *Foundations of Elementary School Arithmetic.* Chicago: Scott Foresman and Co., 1965.

An excellent reference with proper balance between a precise formulation of mathematical concepts and an informal presentation of the content. In the chapter on relations, the study of ordered number pairs is begun in the context of rate pairs. In a later chapter the study of ordered number pairs is continued in the context of fractions. A set of equivalent fractions is called a *rational number of arithmetic.* The various numeral forms are treated under computation involving rational numbers.

Wilhelms, Fred T., *Evaluation as Feedback and Guide.* Washington: Association for Supervision and Curriculum Development, NEA, 1967.

Stresses the idea that the feedback from evaluation controls the next steps and should be based on the objectives.

Wisner, Robert, *A Panorama of Number.* Glenview, Ill.: Scott Foresman and Co., 1965.

The author has employed a unique writing style which is interesting. Many interesting observations are made concerning prime numbers.

U. S. Department of Health, Education and Welfare. *Elementary Arithmetic and Learning Aids.* Washington, D. C. U. S. Government Printing Office, 1965.

This book has descriptions of aids to make and use in elementary mathematics.

INSTRUCTIONAL AIDS

Bags, plastic-assorted sizes
Beads to string
Boxes, assorted sizes
Cards, assorted colors
Cards, decks of Old Maid, Rook, etc.
Chalk, assorted colors
Coins, real or play
Clocks, play
Clothes pins
Compasses
Construction paper, assorted colors
Counters (bottle caps, cardboard discs)
Crayons
Cubes, assorted colors
Dowel rods
Egg cartons
Elastic thread, assorted colors
Flannel board and flannel cutouts
Geoboard
Graph paper
  bulletin board size
  individual size
Logic blocks
Macaroni shells
Maps
Meter sticks
Mirror cards
Multibase blocks
Nails, one size
Number lines
  walk-on (first grade)
  chalkboard size
  individual size
  whole numbers
  fractions
  integers
Objects
  small, for sets
  different colored
Overhead projector
  transparency paper
Paper clips
Paper cups
Paper roll
Paper, squared
  assorted sizes
Pegboard and pegs
Place value charts
Polyhedra models, regular
  made of cardboard or plastic
Ribbon, assorted colors
Rubber bands, assorted sizes
Rulers, foot
  unmarked
  marked in inches
  marked in fractional parts of inches
Sampling box
Sampling ladle
Scales
Scissors
Spinners
  two-colored
  more than two colored
Sticks for bundling
String
Thermometers
  Fahrenheit
  Centigrade (Celsius)
Three-dimensional shapes with flat surfaces
Two-dimensional shapes for "tiling"
Spoons, plastic
Sugar cubes
Tape measures, including steel tapes
Tea
Washers or counting rings of one size
Yardsticks
Yarn, knitting, assorted colors.

GLOSSARY FOR TEACHERS

GLOSSARY FOR TEACHERS

The purpose of the glossary is for clarification of terms for the teacher and is not to be used for pupils. Each teacher should have a reputable student mathematics dictionary for use by the pupils.

ABSCISSA If the ordered pair of numbers (a,b) are the coordinates of a point of a graph, the number a is the abscissa.

ABSOLUTE ERROR One-half the smallest marked interval on the scale being used.

ABSOLUTE VALUE The absolute value of the real number a is denoted by $|a|$. If $a > 0$ then $|a| = a$ and if $a < 0$, $|a| = {}^{-}a$. On the number line absolute value is the distance of a point from zero.

ACCURACY The accuracy of a measurement depends upon the relative error. It is directly related to the number of significant digits in the measured quantity.

ADDITIVE IDENTITY The number I in any set of numbers that has the following property: $I + a = a$ for all a in the set. The symbol for the identity is usually 0; in the complex numbers it is $0 + 0i$, and in some systems bears no resemblance to zero.

ADDITIVE INVERSE For any given number a in a set of numbers the inverse, usually designated by $(-a)$ is that number which when added to a will give the additive identity.

ALGEBRAIC EXPRESSION An algebraic expression may be a single numeral or a single variable; or it may consist of combinations of numbers and variables, together with symbols of operation and symbols of grouping.

ALGORITHM (ALSO ALGORISM) Any pattern of computational procedure.

AMPLITUDE The amplitude of a trigonometric function is the greatest absolute value of the second coordinates of that function. For a complex number represented by polar coordinates the amplitude is the angle which is the second member of the pair.

ANGLE The set of all points on two rays which have the same end-point. The end-point is called the *vertex* of the angle, and the two rays are called the *sides* of the angle.

ANGULAR VELOCITY The amount of rotation per unit of time.

APPROXIMATE MEASURE Any measure not found by counting.

APPROXIMATION The method of finding any desired decimal representation of a number by placing it within successively smaller intervals.

ARC If A and B are two points of a circle with P as center, the arc $\widehat{AB}$ is the set of points in the interior of $\angle APB$ on the circle and on the angle.

AREA OF A SURFACE Area measures the amount of surface.

ARGAND DIAGRAM Two perpendicular axes, one which represents the real numbers, the other the imaginary numbers thus giving a frame of reference for graphing the complex numbers. These axes are called the real axis and the imaginary axis.

ARITHMETIC MEANS The terms that should appear between two given terms so that all the terms will be an arithmetic sequence.

ARITHMETIC SEQUENCE (ALSO PROGRESSION) A sequence of numbers in which there is a common difference between any two successive numbers.

ARITHMETIC SERIES The indicated sum of an arithmetic sequence.

ARRAY A rectangular arrangement of elements in rows and columns.

ASSOCIATIVE PROPERTY A basic mathematical concept that the order in which certain types of operations are performed does not affect the result. The laws of addition and multiplication are stated as $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$.

ASYMMETRIC Having no point, line or plane of symmetry.

AVERAGE A measure of central tendency. See mean, median and mode.

AXIS OF SYMMETRY A line is called an axis of symmetry for a curve if it separates the curve into two portions so that every point of one portion is a mirror image in the line of a corresponding point in the other portion.

BASE The first collection in a number series which is used as a special kind of one. It is used in combination with the smaller numbers to form the next number in the series. In the decimal system of numeration, eleven, which is one more than the base of ten, literally means ten and one. Twenty means two tens or two of the base. In an expression such as a^n, the quantity a is called the *base* and n the exponent.

BASE OF A NUMERATION SYSTEM The base of a numeration system is the number of units in a given digits place which must be taken to denote one in the next higher place.

BASE TEN A system of numeration or a place-value arithmetic using the value of ten as its base value.

BASIC FACTS The name given to any operational table in a base of place-value arithmetic, as, basic addition tables, subtraction tables, multiplication tables, division tables, power tables, logarithmic tables. Basic addition facts include all addition facts in which neither of the addends exceeds 9. Basic subtraction facts include all the subtractions facts which correspond to all basic addition facts. The products formed by ordered pairs composed of the numbers 0 through 9 are called basic multiplication facts. Basic division facts include all the division facts which correspond to the basic multiplication facts except for $a \times b = c$ where $b \neq 0$.

BETWEENNESS B is between A and C if A, B and C are distinct points on the same line and $AB + BC = AC$.

BIAS When the method of selecting samples does not satisfy the condition that every possible sample that can be drawn has an equal chance of being selected, the sampling process is said to be biased.

BINARY OPERATION An operation involving two numbers such as addition; similarly, a unary operation involves only one number as "the square of."

BINARY NUMERATION SYSTEM A system of notation with base two. It requires only two symbols, 0 and 1.

BOUNDED A point set S is bounded if and only if there is a circle (or sphere in 3-space) such that S lies entirely in the interior of that circle (sphere).

CARDINAL NUMBER If two sets can be put in one-to-one correspondence with each other, they are said to have the same cardinal number. A whole number which answers the question of how many in a given finite set is called the cardinal number of a set.

CARTESIAN COORDINATES In a plane, the elements of ordered pairs which are distances from two intersecting lines. The distances from one line is measured along a parallel to the other line.

CARTESIAN PRODUCT The Cartesian product of two sets A and B, written $A \times B$ and read "A cross B" is the set of all ordered pairs (x, y) such that $x \in A$ and $y \in B$.

CELL The union of a simple closed surface and its interior.

CENTRAL TENDENCY A measure of the trend of occurrences of an event.

CHECK To verify the correctness of an answer or solution. It is not to be confused with *prove*.

CIRCLE The set of points in a given plane each of which is at a given distance from a given point of the plane. The given point is called the *center*, and the given distance is called the *radius*.

CIRCULAR FUNCTION A function which associates a unique point with each arc of a unit circle (as measured from a fixed point of the circle). The sine function associates with the measure of an arc the ordinate of its companion point and the cosine the abscissa of the point.

CIRCUMFERENCE The length of the closed curved line which is the circle.

CLOSED CURVE (SIMPLE) A path which starts at one point and comes back to this point without intersecting itself represents a *simple* closed curve.

CLOSURE, PROPERTY OF A set is said to have the property of closure for any given operation if the result of performing the operation on any two members of the set is a number which is also a member of the set.

COLLECTION Elements or objects united from the viewpoint of a certain common property; as collection of pictures, collection of stamps, numbers, lines, persons, ideas.

COMBINATION A combination of a set of objects is any subset of the given set. All possible combinations of the letters a, b and c are a, b, c, ab, ac, bc, abc.

COMMUTATIVE PROPERTY A basic mathematical concept that the order in which certain types of operations are performed does not affect the result. Addition is commutative, for example, $2 + 4 = 4 + 2$. Multiplication is commutative, for example, $2 \times 4 = 4 \times 2$.

COMPASS (OR COMPASSES) A tool used to construct arcs and circles.

COMPLEX FRACTION A fraction that has one or more fractions in its numerator or denominator.

COMPLEX NUMBER Any number of the form $a + bi$ where a and b are real numbers and $i^2 = -1$.

COMPOSITE NUMBER A counting number which is divisible by a smaller counting number different from 1.

CONGRUENT Two configurations which are such that when superimposed any point of either one lies on the other. Two figures having the same size and shape.

CONJUGATE COMPLEX NUMBERS The conjugate of the complex number $a + bi$ is the complex number $a - bi$.

CONJUNCTION A statement consisting of two statements connected by the word *and*. An example is $x + y = 7$ *and* $x - y = 3$. The solution set for a conjunction is the intersection of the solution sets of the separate statements.

CONDITIONAL EQUATION An equation that is true for only certain values of the variable, for example, $x + 3 = 7$.

CONIC, CONIC SECTION The curves which can be obtained as plane sections of a right circular cone.

CONSISTENT SYSTEM A system whose solution set contains at least one member.

CONSTANT A particular member of a specified set.

COORDINATE PLANE A plane whose points are named by ordered pairs of numbers which measure the distances from two intersecting lines. Each distance is measured from one line along a parallel to the other line.

COTERMINAL ANGLES Two angles which have the same initial and terminal sides but whose measures in degrees differ by 360 or a multiple of 360.

COUNTABLE In set theory, an infinite set is countable if it can be put into one-to-one correspondence with the natural numbers.

COUNTING NUMBERS The counting numbers are 1, 2, 3, 4, . . .

CONVERGENT SEQUENCE A sequence that has a limit.

DECIMAL EXPANSION A digit for every decimal place.

DECIMAL FRACTION A decimal fraction is a fraction in which the denominator is a power of ten. The fractions $\frac{3}{10}$, $\frac{4}{100}$, $\frac{125}{1000}$, etc. are decimal fractions but $\frac{2}{3}$ is not.

DEDUCTIVE REASONING The process of using previously assumed or known statements to make an argument for new statements.

DEGREE In angular measure, a standard unit that is $\frac{1}{90}$ of the measure of a right angle. In arc measure, one of the 360 equal parts of a circle.

DEGREE OF A POLYNOMIAL The general polynomial $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x^1 + a^n$ is said to be of degree n if $a_0 \neq 0$.

DENOMINATOR The lower term in a fraction. It names the number of equal parts into which a number is to be divided.

DEPENDENT LINEAR EQUATIONS Equations that have the same solution set.

DEVIATION The difference between the particular number and the average of the set of numbers under consideration is the deviation.

DIFFERENCE The answer or result of a subtraction. Thus, $8 - 5$ is referred to as a difference, not as a remainder.

DIGIT Any one of the ten symbols used in our numeration system – 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (from the Latin, "digitus," or "finger").

DIHEDRAL ANGLE The set of all points of a line and two non-coplanar half-planes having the given line as a common edge. The line is called the *edge* of the dihedral angle. The *side* or *face* consists of the edge and either half-plane.

DIRECT VARIATION The number y varies directly as the number x if $y = kx$ where k is a constant.

DISC The union of a simple closed curve in a plane and its interior.

DISCRIMINANT The discriminant of a quadratic equation $ax^2 + bx + c = 0$ is the number $b^2 - 4ac$.

DISJUNCTION A statement consisting of two statements connected by *or*, for example, $x + y = 7$ *or* $x - y = 3$ The solution set of disjunction is the union of the solution sets of the separate statements.

DISTRIBUTIVE PROPERTY Links addition and multiplication. Examples of distributive property are as follows.

$3 \times 14 = 3\,(10 + 4) = (3 \times 10) + (3 \times 4) = 30 + 12 = 42.$

$4 \times 3\frac{1}{2} = 4\,(3 + \frac{1}{2}) = (4 \times 3) + (4 \times \frac{1}{2}) = 12 + 2 = 14.$

$a\,(b + c) = ab + ac.$

DIVERGENT SEQUENCE A sequence that is not convergent.

DIVISIBLE An integer a is divisible by an integer b is and only if there is some integer c such that $b \times c = a$.

DIVISION The inverse of multiplication. The process of finding how many times one quantity or number is contained in another. For any real numbers a and b, $b \neq 0$, $a \div b$ means a multiplied by the reciprocal of b. Also, $a \div b = c$, if and only if $a = bc$.

DOMAIN OF A VARIABLE The set of all values of a variable is sometimes called its domain.

DUODECIMAL NUMERATION SYSTEM A system of notation with base twelve. It requires twelve symbols – 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E.

ELEMENTS In mathematics the individual objects included in a set.

EMPTY SET The set which has no elements. The symbol for this set is ϕ or $\{\}$.

END POINT The point on a line from which a ray extends is called the end point of the ray.

EQUALITY The relation *is equal to* denoted by the symbol " =."

EQUATION A sentence (usually expressed in symbols) in which the verb is "is equal to."

EQUIVALENT EQUATIONS Equations that have the same solution set.

EQUIVALENCE RELATION Any relation which is reflexive, symmetric and transitive, for example, reflexive: $a = a$; symmetric: if $a = b$ then $b = a$ and transitive: if $a = b$ and $b = c$, than $a = c$.

EQUIVALENT FRACTIONS Two fractions which represent the same number.

EQUILATERAL TRIANGLE A triangle whose sides have equal length.

ESTIMATE A quick and frequently mental operation to ascertain the approximate value of an involved operation.

EVEN NUMBER An integer that is divisible by 2. All even numbers can be written in the form $2n$, where n is an integer.

EXPANDED EXPONENTIAL FORM The expanded exponential form of a numeral is the form in which the additive, multiplicative and place value properties of a numeration system are explicitly indicated. The value of each place is written in exponential form, for example, $365 = 3(10^2) + 6(10^1) + 5(10^0)$.

EXPONENT In the expressionon a^n the number n is called an exponent. If n is a positive integer it indicates how many times a is used as a factor.

$$a^n = \underbrace{a \times a \times \ldots \times a}_{n \text{ factors}}$$

Under other conditions exponents can include zero, negative integers, rational and irrational numbers.

EXPONENTIAL EQUATION An equation in which the independent variable appears as an exponent.

EXPONENTIAL FUNCTION A function defined by the exponential equation $y = a^x$ where $a > 0$.

EXTRANEOUS ROOTS Those roots in the solution set of a derived equation which are not members of the solution set of the original equation.

EXTRAPOLATING Estimating the value of a function greater than or less than the known values; making inferences from data beyond the point strictly justified by the data.

FACTOR The integer m is a factor of the integer n if $m \times q = n$ where q is an integer. The polynomial $R(x)$ is a factor of the polynomial $P(x)$ if $R(x) \times Q(x) = P(x)$ where $Q(x)$ is a polynomial. *Factorization* is the process of finding the factors.

FACTORIAL The expression "$n!$" is read n factorial. $n! = n(n-1)(n-2)\ldots 2 \times 1$.

FIGURATE NUMBERS Figurate numbers include the numbers more commonly referred to as square numbers, triangular numbers, etc.

FINITE SET In set theory, a set which is not infinite.

FRACTION A symbol $\frac{a}{b}$ where a and b are numbers, with b not zero.

FREQUENCY A collection of data is generally organized into several categories according to specified intervals or subcollections. A frequency is the number of scores or measures in a particular category.

FREQUENCY, CUMULATIVE The sum of frequencies preceding and including the frequency of measures in a particular category is the cumulative frequency.

FREQUENCY DISTRIBUTION A tabulation of the frequencies of scores or measures in each of the categories of data.

FUNCTION A relation in which no two of the ordered pairs have the same first member. Also, alternately, a function consists of (1) a set A called the domain, (2) a set B called the range, (3) a table, rule, formula or graph which associated each member of A with exactly one member of B.

FUNDAMENTAL THEOREM OF ARITHMETIC Any positive integer greater than one may be factored into primes in essentially one way. The order of the primes may differ but the same primes will be present. Alternately, any integer except zero can be expressed as a unit times a product of its positive primes.

FUNDAMENTAL THEOREM OF FRACTIONS If the numerator and denominator are both multiplied (or divided) by the same non-zero number, the result is another name for the fraction.

GEOMETRIC MEANS The terms that should appear between two given terms so that all of the terms will form a geometric sequence.

GEOMETRIC SEQUENCE A sequence in which the ratio of any term to its predecessor is the same for all terms.

GEOMETRIC SERIES The indicated sum of a geometric sequence.

GRAPH A pictorial representation of a set of points associated with a relation which involves one or more variables.

GREATEST INTEGER FUNCTION Is defined by the rule $f(x)$ is the greatest integer not greater than x. It is usually denoted by the equation $f(x) = [x]$

GREATEST LOWER BOUND A lower bound a of a set S of real numbers is the greatest lower bound of S if no lower bound of S is greater than a.

HARMONIC MEAN A number whose reciprocal is the arithmetic mean between the reciprocals of two given numbers.

HEMISPHERE If a sphere is divided into two parts by a plane through its center, each half is called a hemisphere.

HISTOGRAM A bar graph representing a frequency distribution. The base of each of the contiguous rectangular bars is the range of measures within a particular category, and the height of each of the bars is the frequency of measures in the same category.

IDENTITY, IDENTICAL EQUATION A statement of equality, usually denoted by $\equiv$ which is true for all values of the variables. The values of the variable which have no meaning are excluded, for example, $(x + y)^2 = x^2 + 2xy + y^2$.

INCONSISTENT SYSTEM OF EQUATIONS A system whose solution set is the empty set.

INDEPENDENT EVENTS Two events are said to be independent if the occurrence of one does not affect the probability of occurrence of the other.

INDEPENDENT SYSTEM OF EQUATIONS A system of equations that are not dependent.

INDEX The number used with a radical sign to indicate the root. ($\sqrt[3]{\ }$ In this example the index is three.) If no number is used, the index is two.

INDUCTIVE REASONING The process of drawing a conclusion by observing what happens in a number of particular cases.

INEQUALITY The relation in which the verb is one of the following– is not equal to, is greater than or is less than, denoted by the symbols $\neq$, $>$, $<$; respectively.

INFINITE DECIMAL (Also non-terminating) A decimal representation that has an unending string of digits to the right of the decimal point.

INFINITE REPEATING DECIMAL A decimal representation containing a finite block of digits which repeats endlessly.

INFINITE SET In set theory, a set which can be placed in one-to-one correspondence with a proper subset of itself.

INTEGER Any one of the set of numbers which consists of the natural numbers, their opposites and zero.

INTERCEPT If the points whose coordinates are $(a,0)$ and $(0,b)$ are points on the graph of an equation, they are called intercepts. The point whose coordinates are $(a,0)$ is the x-intercept, and the point whose coordinates are $(0,b)$ is the y-intercept.

INTERPOLATION The process of estimating a value of a function between known values other than by the rule of the table of the function.

INTERSECTING LINES Two or more lines that pass through a single point in space.

INTERSECTION OF SETS If A and B are sets, the intersection of A and B, denoted by $A \cap B$, is the set of all elements which are members of both A and B.

INVERSE OF AN OPERATION That operation which, when performed after a given operation, annuls the given operation. Subtraction of a quantity is the inverse of addition of that quantity. Addition is likewise the inverse of subtraction.

INVERSE FUNCTION If f is a given function then its inverse is the function (provided f is one-to-one) formed by interchanging the range with domain. The symbol for inverse of f is f^{-1}.

INVERSE VARIATION The number y is said to vary inversely as the number x if $x \times y = k$ where k is a constant.

IRRATIONAL EQUATION An equation containing the variable or variables under radical signs or with fractional exponents.

IRRATIONAL NUMBER An irrational number is not a rational number. That is, it is a number that cannot be expressed in the form $\frac{a}{b}$ where a and b are integers. The union of the set of rationals and the set of irrationals is the set of real numbers.

JOINT VARIATION A quantity varies jointly as two other quantities if the first is equal to the product of a constant and the other two, for example, y varies jointly as x and w if $y = kxw$.

LATTICE POINTS An array of points named by ordered pairs.

LEAST COMMON MULTIPLE The least common multiple of two or more numbers is the common multiple which is a factor of all the other common multiples.

LEAST UPPER BOUND An upper bound b of a set S of real numbers is the least upper bound of S if no upper bound of S is less than b.

LINEAR EQUATION An equation in standard form in which the sum of the exponents of the variable in any term equals one.

LINEAR MEASURE A measure used to determine length.

LOGARITHM The exponent that satisfies the equation $b^x = n$ is called the logarithm of n to the base b for any given positive number n.

LOWER BOUND A number a is called a lower bound of set S of real numbers if $a \leqslant x$ for every $x \in S$.

MAGIC SQUARE A square of numbers possessing the particular property that the sums in each row, column and diagonal are the same.

MATRIX A rectangular array of numbers.

Example

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$$

MEAN In a frequency distribution, the sum of the n measures divided by n is called the mean.

MEASUREMENT A comparison of the capacity, length, etc., of a thing to be measured with the capacity, length, etc., of an agreed upon unit of measure. Non-standard units are used before standard units of measure are introduced.

MEDIAN In a frequency distribution, the measure that is in the middle of the range when elements are ranked from highest to lowest is called the median. In geometry, a median of a triangle is a line joining a vertex to the midpoint of the opposite side.

MODE In a frequency distribution, the interval in which the largest number of measures fall is called the mode. Alternately, in a frequency distribution, the measure which appears most frequently in the group is called the mode. There may be more than one mode in a set of measures.

MODULO ARITHMETIC For a given positive integer n, *modulo* n is obtained by using the integers $0 \ldots n-1$ and defining addition and multiplication by letting the sum of $a + b$ and the product of $a\ b$ be the remainder after division by n of the ordinary sum and product of a and b. (This is often called clock arithmetic.)

MODULUS A statement of the type x is congruent to y modulus (or modulo) w, w is the modulus of the congruency. If 2 is congruent to 9, then the modulus is 7.

MULTIPLE If a and b are integers such that $a = b \times c$ where c is an integer, then a is said to be a multiple of b.

MULTIPLICATION A short method of adding like groups or addends of equal size. It may be illustrated on a number line by counting forward by equal groups.

MULTIPLICATIVE INVERSE The multiplicative inverse of a non-zero number a is the number b such that $a \times b = 1$. It is usually designated by $\frac{1}{a}$ or a^{-1}.

MUTUALLY DISJOINT SETS Two sets having no elements in common.

MUTUALLY EXCLUSIVE EVENTS Events which cannot occur simultaneously. Mutually exclusive subsets are subsets that are disjoint.

NATURAL NUMBERS Any of the set of counting numbers. The set of natural numbers is an infinite set; it has a smallest member (1) but no largest.

NULL SET A set containing no elements. It is sometimes called an empty set. The symbol for the null set is ϕ or $\{\ \}$.

NUMBER SYSTEM A number system consists of a set of numbers, two operations defined on the set, the properties belonging to the set and a definition for equivalence between any two members of the set.

NUMERATION SYSTEM A coding system for recording numerals. Modern systems of numeration are characterized by a set of symbols, or digits, a place value scheme and a base.

NUMERATOR The upper term in a fraction.

NUMERAL A written symbol for a number, for example, several numerals for the same number are 8, VIII, $7 + 1$, $10 - 2$, $\frac{16}{2}$

OBTUSE ANGLE If the degree measure of an angle is between 90 and 180, the angle is called an obtuse angle.

ODD NUMBER An odd number is an integer that is not divisible by 2; any number of the form $2n + 1$, where n is an integer.

ONE-TO-ONE CORRESPONDENCE A pairing of the members of a set A with members of a second set B such that each member of A is paired with exactly one member of B, and each member of B is paired with exactly one member of A.

OPEN SENTENCE An open sentence is a sentence involving one or more variables, and the question of whether it is true cannot be decided until definite values are given to the variables, for example, $x + 5 = 7$.

ORDERED N-TUPLE A linear array of numbers $(a_1, a_2, a_3, \ldots, a_n)$ such that a_1 is the first number, a_2 is the second number, a_3 is the third number, ... and a_n is the nth number.

ORDERED PAIR A pair of numbers (a,b) where a is the first member and b is the second member of the pair.

ORDINAL NUMBER A number that denotes order of the members in a set.

ORDINATE If an ordered pair of numbers (a,b) are coordinates of a point P, b is called the ordinate of P.

PARALLEL LINES Two straight lines in a plane that do not intersect however far extended.

PARALLELOGRAM A quadrilateral whose opposite sides are parallel.

PARAMETER An arbitrary constant or a variable in a mathematical expression, which distinguishes various specific cases.

PARTIAL PRODUCT Used in elementary arithmetic with regard to the written algorithm of multiplication. Each digit in the multiplier produces one partial product. The final product is then the sum of the partial products.

PARTIAL QUOTIENT In long division, any of the trial quotients that must be added to obtain the complete quotient.

PERIMETER The sum of the measures of the sides of a polygon. The measure of the outer boundary of a polygon.

PERIOD The number of digits set off by a comma in an integer or the integral part of a mixed decimal. In a repeating decimal the period is the sequence of digits that repeats.

PERIODIC FUNCTION A function from R to R, where R is the set of real numbers, is called periodic if, and only if, $f(x)$ is not the same for all x and there is a real number p such that $f(x+p) = f(x)$ for all x in the domain of f. The smallest positive number p for which this holds is called the period of the function.

PERMUTATION A permutation is an ordered arrangement of all or part of the members in a set. All possible permutations of the letters a, b and c are a, b, c, ab, ac, ba, bc, ca, cb, abc, acb, bac, bca, cab, cba.

PERPENDICULAR A line is perpendicular to a ray if and only if the end point of the ray is the only point of intersection of the two and the two angles formed are congruent.

PLACE VALUE The value of a numeral is dependent upon its position. In the number 324, for example, each digit has a place value 10 times that of the place value of the digit to its immediate right.

PLANE ANGLE Through any point on the edge of a dihedral angle pass a plane perpendicular to the edge intersecting each side in a ray. The angle formed by these rays is called the plane angle of the dihedral angle.

POLAR COORDINATES An ordered pair used to represent a complex number. The first member of the pair is the number of units in the radius vector, and the second member is the angle of rotation of the radius vector.

POLYGON A simple closed curve which is the union of line segments is called a polygon.

POLYHEDRON A solid bounded by plane polygons. The bounding polygons are the *faces*, the intersections of the faces are the *edges* and the points where three or more edges intersect are the *vertices*.

POLYNOMIAL An algebraic expression of the form $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ sometimes designated by the symbol $P(x)$.

POLYNOMIAL EQUATION A statement that $P(x) = 0$.

POLYNOMIAL FUNCTION A function defined by a polynomial equation or $f: x \rightarrow P(x)$.

PRECISION The precision of a measurement is inversely related to the absolute error. Thus the smaller the absolute error, the greater the precision.

PRIME NUMBER A counting number other than one, which is divisible only by itself and one.

PRISM If a polyhedron has two faces parallel and its other faces in the form of parallelograms, it is called a prism.

PROBABILITY The numerical measure of the likelihood of an event is called the probability of the event. It is a rational number p such that $0 \leq p \leq 1$.

PROPER SUBSET A subset R is a proper subset of a set S if R is a subset of S and $R \neq S$.

PURE IMAGINARY A complex number $a + bi$ in which $a = 0$ and $b \neq 0$.

PYRAMID A polyhedron, one of whose faces is a polygon of any number of sides and whose other faces are triangles having a common vertex.

QUADRANTAL ANGLE If the terminal side of an angle with center at the origin coincides with a coordinate axis, the angle is called a quadrantal angle.

QUADRILATERAL A polygon formed by the union of 4 line segments.

QUINARY SYSTEM OF NUMERATION A system of notation with the base 5. It requires only five symbols or digits—0, 1, 2, 3, 4.

RADIAN MEASURE Angular measure where the unit is an angle whose arc on a circle with center at vertex of angle is equal in length to the radius of the circle.

RADIUS Any line segment with endpoint at the center of a circle and the other endpoint on the circle is called a radius of the circle.

RADIUS VECTOR A line segment with one end fixed at the origin on the cartesian plane and rotating from an initial position along the positive x-axis so that its free end point generates a circle.

RANGE (STATISTICS) The range of the set of numbers is the difference between the largest and smallest numbers in a set.

RANGE (OF A FUNCTION) The set of all elements assigned to the elements of the domain by the rule of the function.

RATE PAIR An ordered pair of counting numbers which expressed a rate relation – e.g., a rate of exchange. In general, a rate pair $\frac{a}{b}$, where a and b are counting numbers, expresses a ratio of the number of elements in one set to the number of elements in a second set.

RATIONAL EXPRESSION A rational expression is a quotient of two polynomials or in symbols $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.

RATIONAL NUMBER If a and b are whole numbers with b not zero, the number represented by the fraction $\frac{a}{b}$ is called a rational number.

RATIONAL NUMBERS OF ARITHMETIC In the elementary school, one generally defines a set of equivalent fractions to be a rational number. Alternatively, a rational number is an equivalence class of ordered pairs of integers a and b, $b \neq 0$.

RAY Let A and B be points on a line. Then ray $\overrightarrow{AB}$ is the set which is the union of the segment $\overline{AB}$ and the set of all points C for which it is true that B is between A and C. The point A is called the *end-point* of $\overrightarrow{AB}$.

RECIPROCAL Multiplicative inverse.

RECIPROCAL FUNCTION Pairs of functions in the set of real numbers whose product is 1, for example, $(\sin \phi)(\csc \phi) = 1$.

RECTANGLE A parallelogram with right angles.

REFERENCE TRIANGLE For any angle on the Cartesian plane with vertex at the origin, the triangle formed by the radius vector, its projection on the x-axis and a line drawn from the end of the radius vector perpendicular to the x-axis is called the reference triangle.

REFLECTION IN A LINE A point P has a mirror image P' in the line $\overleftrightarrow{AB}$ if P, P' and $\overleftrightarrow{AB}$ all lie in the same plane with P [illegible] opposite sides of $\overleftrightarrow{AB}$ and if the perpendicular distances PO and $P'O$ to the point O in $\overleftrightarrow{AB}$ are equal.

REFLEXIVE PROPERTY If a is any element of a set and if R is a relation on the set such that aRa for all a, then R is reflexive.

REGION The union of a simple closed curve and its interior.

RELATED ANGLE For any angle on the Cartesian plane, the related angle is the angle in the reference triangle formed by the radius vector and x-axis.

RELATION A relation from set A to set B (where A and B may represent the same set) is any set of ordered pairs (a,b) such that a is a member of A and b is a member of B.

RELATIVE ERROR Ratio of the absolute error to the measured value.

RELATIVE FREQUENCY The relative frequency is the frequency of a given category divided by the total number of measures in the category.

RELATIVELY PRIME Two integers are relatively prime if they have no common factors other than +1 or −1; two polynomials are relatively prime if they have no common factors except constants.

REPEATING DECIMAL A decimal numeral which never ends and which repeats a sequence of digits. It is indicated in this manner – 0.333 ... or $0.\overline{142857}$.

RESOLUTION OF VECTORS The process of finding the vertical and horizontal components.

RESTRICTED DOMAIN Domain of a function or relation from which certain numbers are excluded for reasons such as division by zero is not permitted and need for the inverse of a function to be a function.

RIGHT ANGLE Any of the four angles obtained at the point of intersection of two perpendicular lines. The angle made by two perpendicular rays. Its measure is 90 degrees.

RIGHT TRIANGLE A triangle with one right angle.

ROUNDING OFF Replacing digits with zero's to a certain designated place in a number with the last remaining digit being increased or decreased under certain specified conditions.

SAMPLE SPACE The set of all possible outcomes of an experiment.

SCALAR In physical science, a quantity having magnitude but no direction. In a study of mathematical vector, any real number.

SCALE A system of marks in a given order and at fixed intervals. Scales are used on rulers, thermometers and other measuring instruments and devices as an aid in measuring quantities.

SCIENTIFIC NOTATION A notation generally used for very large or very small numbers in which each numeral is changed to the form $a \times 10^k$ where a is a real number containing at most three significant digits such that $1 \leqslant a < 10$ and k is any integer.

Example

$6{,}708{,}345 = 6.71 \times 10^6$

$.000000052 = 5.2 \times 10^{-8}$

SEGMENT For any two points A and B, the set of points consisting of A and B and all points between A and B is the line segment determined by A and B. The segment is a geometrical figure while the distance is a number which tells how far A is from B.

SEQUENCE An ordered arrangement of numbers.

SERIES The indicated sum of a sequence.

SET A collection of particular things, as a set of numbers between 3 and 5, the set of points on the segment of a line or within a circle.

SET BUILDER NOTATION To describe the members of a very large or infinite set, it is often helpful to denote the set [illegible] members as in this example – $\{x \mid x \in R \text{ and } 0 \leqslant x \leqslant 1\}$, read "The set of all x such that x is a member of the set R of rational numbers and x is greater than or equal to 0 and less than or equal to 1." The symbol device, $\{x \mid x \ldots\}$, read "the set of all x such that x..." is called set builder notation.

SIGNIFICANT FIGURE Any digit or any zero in a numeral not used for placement of the decimal point, for example, 703,000; .0056; 5.00.

SIMILAR Two geometric figures are similar if one can be made congruent to the other by using a transformation of similitude if one is a magnification or reduction of the other.

SKEW LINES Two lines which are not coplanar are said to be skew.

SLOPE The slope of a given segment (P_1P_2) is the number m such that $m = \frac{y_2 - y_1}{x_2 - x_1}$ where P_1 is the ordered pair (x_1, y_1) and P_2 is the ordered pair (x_2, y_2).

SOLID Any simple closed surface; the term is usually used with reference to polyhedra (rectangular solids, pyramids); cylinders, cones and spheres.

SOLUTION SET The truth set of an equation or a system of equations.

SPHERE The set of all points in space each of which is at a given distance from a given point. The given point is called the *center* of the sphere and the given distance is called the *radius*.

SQUARE A quadrilateral formed by four line segments of equal length which meet at right angles.

STANDARD DEVIATION The square root of the arithmetic mean of the squares of the deviations from the mean.

STATISTIC An estimate of a parameter obtained from a sample.

STATISTICS The concepts, measures and techniques related to methods of obtaining, organizing and analyzing data is included in statistics.

SUBSET A set contained within a set; a set whose members are members of another set. The fact that R is a subset of S is indicated by $R \subset S$.

SUBTRACTION To subtract the real number b from the real number a, add the opposite (additive inverse of b to a. $a - b = a + (-b)$. Also, $a - b = c$ if and only if $a = c + b$.

SUCCESSOR The successor of the integer a is the integer $a + 1$.

SUMMATION NOTATION The symbol $\sum\limits_{k=n}^{n} a_k$. The symbol Σ, the Greek letter "sigma," corresponds to the first letter of the word "sum" and is used to indicate the summing process. The k and n represent the upper and lower indexes and indicate that the summing begins with the kth term and includes the nth term, for example,

$$\sum_{k=2}^{5} a_k = a_2 + a_3 + a_4 + a_5.$$

When the summation includes infinitely many terms is it written $\sum\limits_{k}^{\infty} a_k$. In this case there is no last term a because ∞ is not a number. The symbol ∞ is used to indicate that the summation is infinite.

SYMMETRIC PROPERTY If a and b are any elements of a set and if R is a relation on the set such that aRb implies bRa, then the relation is said to have the symmetric property.

TERM In a phrase which has the form of an indicated sum, $A + B$, A and B are called *terms* of the phrase.

TERMINATING DECIMAL (Also finite decimal) A decimal representation that contains a finite number of digits.

TOPOLOGY A branch of mathematics which is the study of properties of point sets which are preserved.

TRANSITIVE PROPERTY If a, b, and c are any elements of a set and if R is a relation on the set such that aRb and bRc imply aRc, then the relation is said to have the transitive property.

TRAPEZOID A quadrilateral with at least two parallel sides.

TRIANGLE If A, B and C are three non-collinear points in a given plane, the set of all points in the segments having A, B, C as their end points is called a triangle.

UNBOUNDED Not bounded.

UNEQUAL Not equal, symbolized by $\neq$.

UNION OF SETS If A and B are two sets, the union of A and B is the set $A \cup B$ contains all the elements and only those elements that are in A or in B, for example, $A = \{2, 8, 3\}$, $B = \{5, 2, 7, 6\}$ then $A \cup B = \{2, 8, 3, 5, 7, 6\}$.

UNIQUE One and only one.

UPPER BOUND A number b is called an upper bound of a set S of real numbers if $b \geq x$ for every $x \in S$.

VARIABLE A letter used to denote any one of a given set of numbers. Another name for variable is placeholder in an equation, for example, $x + 5 = 7$.

VECTOR In physical science, a quantity having magnitude and direction. In mathematics a vector is a matrix of one row or one column as $(a_1\ b_1\ c_1)$ or

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

VERTEX The point of intersection of two rays.

VOLUME The amount of space occupied by a solid or enclosed within it.

WHOLE NUMBERS The whole numbers are 0, 1, 2, 3, 4, ...

SYMBOLS

$\neq$	is not equal to
$\approx$	is approximately equal to
$>$	is greater than
$\not>$	is not greater than
$<$	is less than
$\not<$	is not less than
$\geqslant$	is greater than or equal to
$\not\geqslant$	is not greater than or equal to
$\leqslant$	is less than or equal to
$\not\leqslant$	is not less than or equal to
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\supset$	is a superset of
$\cong$	is congruent to
$\sim$	is similar to
$\in$	is an element of
$\notin$	is not an element of
U	universal set
S	solution set
$\bar{S}$	complement set

$A \times B$	Cartesian product set of sets A and B
a^{-n}	is interpreted as $\frac{1}{a^n}$ where $a \neq 0$
	is parallel to
$\perp$	is perpendicular to
$\overleftrightarrow{AB}$	straight line containing points A and B
$\overline{AB}$	straight line segment with end points A and B
$\overrightarrow{AB}$	ray from point A through point B
(a,b)	ordered pair a and b
$\{a\}$	set containing element a
$\square, \triangle$	frames, place holders or nonspecified elements
$\phi, \{\ \}$	the empty of null set
$\triangle ABC$	triangle with vertices A, B, and C applies to any polygon
$\angle ABC$	angle with point B as vertex
$\{\square \mid \square > 5\}$	the set of all $\square$ in the universal set such that $\square$ is greater than 5
$a:b$	ratio of a to b
	union of two sets
$\cap$	intersection of two sets